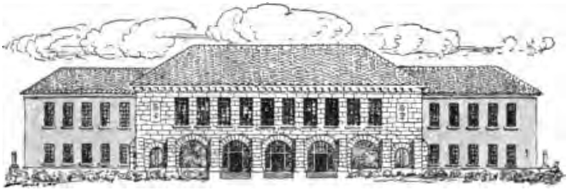
SCHOOL ALGEBRA

BAILEY



HIGH SCHOOL ALGEBRA

BY

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PREFACE

This book, designed for use in high schools and academies, embraces all the topics in algebra usually required for admission to college. The aim has been to combine simplicity so far as practicable with scientific rigor; consequently, many assumptions which, though far from self-evident, are too often taken for granted, are here submitted to proof.

The fundamental principles are stated and proved after the manner of theorems in geometry. Pupils are expected to memorize the statements of the propositions and to demonstrate them in a formal manner at the blackboard. The emphasis is placed upon principles rather than upon examples, under the conviction that if we "take care of the principles the examples will take care of themselves." The mere reading, without memorizing, of statements, or the mere discussion, without reproduction, of demonstrations never makes a definite or a lasting impression. The pupil who solves examples merely by models becomes an imitator instead of a thinker. When tested by the college examiner, he is found to possess many fragmentary and crude ideas, but little that is definite, accurate, and scientific.

The developments (pp. 18, 28, 127, etc.) present the philosophy of numbers with clearness and conciseness. From the very nature of each subject certain needs are seen to grow, and expedients are suggested as means of satisfying these needs. Thus, from the self-evident truth that the whole is equal to all its parts, addition or subtraction is seen to arise as the whole or as a part is required; multiplication or division is seen to arise as the product or as the quotient is required; and so on. Pupils are expected to present these developments as wholes, because power to give full and comprehensive discussions without either questions or suggestions from the teacher is of prime importance.

The same general plan has been followed that characterizes the other books of the series. The path of procedure is always from the known to the related unknown; the solution of every example is traced to its source in one of the fundamental principles; definitions are placed in alphabetical order at the end of the book; the pupil is never allowed to grope in the dark, but is taught to keep constantly in mind the end, to consider carefully the means, and to exercise his judgment.

The pupil who has mastered this treatise has gained a power in close discrimination and logical analysis that will stand him in good stead in practical life, or that will enable him to follow with profit a more advanced course in the science of algebra.

M. A. BAILEY.

CONTENTS

ELEMENTS OF ALGEBRA

NOTATION AND NUMERATION

THE PROVINCE OF ALGEBRA

The province of algebra is to find expedients to satisfy some needs in connection with number that have not been fully satisfied in arithmetic. Just as the problems of arithmetic might be solved by counting alone without the aid of the processes of addition, subtraction, multiplication, and division, so the problems of algebra might be solved by the methods of arithmetic alone; but just as the method of counting is more troublesome than multiplication for the finding of products, so arithmetic is more troublesome than algebra for the solution of certain classes of problems.

Algebra makes its chief advance over arithmetic in the use of letters for numbers, in the association of numbers with signs of direction, and in the nature of its attack upon problems.

In a treatise on algebra it is not necessary to repeat all the definitions, principles, and operations that have been established in arithmetic, because they hold good in both sciences. It remains only to call special attention to those terms whose meanings have been extended, to replace illustrations of principles by proofs, and to apply the operations to complex expressions.

NUMBERS BY LETTERS

PROPOSITION I. AGREEMENT

The first letters of the alphabet, as a, b, c, may be used for known numbers; the last letters, as x, y, z, for unknown numbers.

This expedient satisfies in one particular the need for a briefer notation in the solution of problems.

1. Multiply 62,386 by 12 and divide the result by 3.

WITHOUT LETTERS	WITH LETTERS
62,386	Let $a = 62,386$
12	$12a =$ the product
3)748,632 product	$4a =$ the quotient
249,544 quotient	$249,544 =$ the quotient

At the right, a is used instead of the given number. It is easier to multiply a by 12 than to multiply 62,386 by 12, and easier to divide $12a$ by 3 than to divide 748,632 by 3. Although the method with letters involves the final multiplication of 62,386 by 4, yet this labor is more than offset.

2. If a certain number is multiplied by 6 and the product is divided by 2, the quotient is 72. Find the number.

WITHOUT LETTERS	WITH LETTERS
the number = the number	Let $x =$ the number
6 times the number = the product	$6x =$ the product
3 times the number = the quotient	$3x =$ the quotient
3 times the number = 72	$3x = 72$
the number = 24	$x = 24$

At the right, x is used instead of the required or unknown number. It is easier to say or to write x than *the number;* it is easier to say or to write $6x$, than *6 times the number;* and $3x$, than *3 times the number.*

3. Divide 748,324 by 24 and multiply the result by 12. Represent 748,324 by a.

4. If a certain number (x) is divided by 12 and the quotient is multiplied by 6, the product is 144. Find the number.

This expedient satisfies the need for a briefer notation in the statement and discussion of general truths.

5. By the use of letters for numbers, state the general rule for multiplying fractions.

Ans. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$

$\frac{a}{b}$ represents any fraction, because its numerator a stands for any number, and its denominator b for any number. For the same reason, $\frac{c}{d}$ represents any fraction. The above expression is interpreted: To multiply fractions, multiply the numerators for a new numerator, and the denominators for a new denominator.

6. Interpret the following general statement:

$$(a + b) \times (a - b) = a^2 - b^2$$

Ans. The product of the sum and difference of two quantities is equal to the difference of their squares.

$a + b$ is the sum of any two quantities; $a - b$, their difference; $a^2 - b^2$, the difference of their squares.

By the use of letters for numbers state:

7. To divide fractions, divide the numerators for a new numerator, and the denominators for a new denominator.

8. To divide fractions, invert the divisor and proceed as in multiplication.

9. Multiplying both numerator and denominator by the same number does not change the value of a fraction.

Interpret the following general statements:

10. $\frac{a \times c}{b} = \frac{a}{b} \times c$

11. $\frac{a}{b \times c} = \frac{a}{b} \div c$

12. $\frac{a \div c}{b} = \frac{a}{b} \div c$

13. $\frac{a}{b \div c} = \frac{a}{b} \times c$

14. $\frac{a \times c}{b \times c} = \frac{a}{b}$

15. $\frac{a \div c}{b \div c} = \frac{a}{b}$

NUMBERS WITH DIRECTION

Proposition II. Agreement

The sign, '+,' has an arbitrary meaning; the sign, '–,' denotes the opposite of '+' in the same position.

Thus, if '+' means *add,* '–' means *subtract;* if '+' means *north,* '–' means *south;* if '+' means *up,* '–' means *down.*

This expedient satisfies in another particular the need for a briefer notation.

16. By degrees, count backward from 3° above zero to 3° below zero. What need is felt?

Ans. 3° above zero, 2° above zero, 1° above zero, 0°, 1° below zero, 2° below zero, 3° below zero. The need for a briefer notation is felt.

17. If '+' denotes *above zero,* how may *below zero* be indicated? Write in this way, from 3° above zero to 3° below zero.

Ans. Below zero may be indicated by '–.' $+3°$, $+2°$, $+1°$, $0°$, $-1°$, $-2°$, $-3°$.

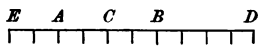
18. A man has \$6, spends \$2 three times and then runs into debt \$2 three times. State his condition after each time.

Ans. \$4 in hand, \$2 in hand, \$0 in hand, \$2 in debt, \$4 in debt, \$6 in debt.

19. If '+' denotes *in hand,* how may *in debt* be indicated? How may each of the conditions be expressed by a briefer notation?

Ans. In debt may be indicated by '–.' +\$4, +\$2, \$0, –\$2, –\$4, –\$6.

20. If '+' means *to the right,* proceed from the point, *A,* on a horizontal line, +4 spaces, then –2 spaces, then +6 spaces, then –10 spaces.



Ans. +4 spaces, *A* to *B*; –2 spaces, *B* to *C*; +6 spaces, *C* to *D*; –10 spaces, *D* to *E*. *E* is –2 spaces from *A*.

Addition and subtraction denote movements in opposite directions. Thus, to add 2¢ to 6¢ is to increase the amount; to subtract 2¢ from 6¢ is to decrease the amount by the same difference. '+' may, therefore, denote addition and '–,' subtraction. In this case, the signs are written before numbers. Thus,

$+2$ means that 2 is to be used once as an addend.
-2 means that 2 is to be used once as a subtrahend.

Multiplication and division denote movements in opposite directions. Thus, to multiply 6¢ by 2 is to increase the amount; to divide 6¢ by 2 is to decrease the amount in the same ratio. '+' may, therefore, denote multiplication and '–' division. In this case, the signs are written above numbers. Thus,

2^{+1} means that 2 is to be used once as a multiplier. (See p. 14.)
2^{-1} means that 2 is to be used once as a divisor. (See p. 14.)

In arithmetic, the sign '+' is always the symbol of addition and the sign '–,' the symbol of subtraction. In algebra, these signs are sometimes so considered, but they are usually regarded as parts of the terms with which they occur.

Thus, in algebra it is best to regard $8+6$ as meaning the sum of $+8$ and $+6$, or the resultant of taking some unit 8 times as an addend and the same unit 6 times more as an addend. It is best to regard $8-6$ as meaning the sum of $+8$ and -6, or the resultant of taking some unit 8 times as an addend and the same unit 6 times as a subtrahend. If the expressions were written $(+8)+(+6)$ and $(+8)+(-6)$, they would mean as just explained; the '+' outside of the parentheses would then be the symbol of addition while the signs within would belong to 8 and to 6.

The algebraic integers, or integers with direction, may be obtained by counting forwards and backwards from 0.

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots$$

It follows that any one of these numbers is smaller than any number farther to the right.

NOTE. The dots indicate that the operation may be continued without limit.

TERMS — COEFFICIENTS

Proposition III. Agreement

A positive coefficient shows how many times the base is used as an addend.

Thus, $+3a = a + a + a$, where a is used three times as an addend.

The expression, $+3a$, is a term; $+3$, the coefficient; a, the base; $+1$, the exponent (p. 14). A term always consists of these three parts, but one or two parts may be understood.

Proposition IV. Theorem

A negative coefficient shows how many times the base is used as a subtrahend.

To prove that $-3a = -a - a - a$

$+3a$ means that a is used 3 times as an addend. Since '$-$' means the opposite of '$+$,' $-3a$ means that a is used 3 times in a sense opposite to an addend, or 3 times as a subtrahend, or $-3a = -a - a - a$.

Hence, the principle, Q.E.D.

Note. Q.E.D. means *which was to be proved*. It is inserted simply to show that the demonstration is finished.

Proposition V. Theorem

*A zero coefficient shows that the base is used the same number of times both as an addend and as a subtrahend; the value of a term with zero coefficient is **0**.*

$$0 = +3 - 3$$
$$0a = (+3 - 3)a = a + a + a - a - a - a = 0$$

In the same way, it may be shown that $0a = (+4 - 4)a$, $(+5 - 5)a$, and so on.

Hence, the principle, Q.E.D.

21. Analyze $4b$. Analyze ax.

Ans. $4b$ is a term; $+4$ is the coefficient; b is the base. It means $b+b+b+b$. ax is a term; $+a$, the coefficient; x, the base; it means $x+x+x+\cdots$ where x is used a times as an addend.

NOTE. A series of periods, $\cdots$, means *and so on.*

22. Analyze $+3c$; $7c$; bc.

23. Analyze $+6a$; $+ac$.

24. If $b=5$, find the value of $4b$; of $6b$; of $8b$; of ab.

25. If $a=6$, find the value of $+6a$; of $+3a$; of $+125a$.

26. Express as concisely as possible that x is taken 5 times as an addend; that 3 is taken 5 times as an addend.

Ans. $5x$; 5×3.

27. Explain the meaning of abc.

Ans. abc means that c is taken b times as an addend, and that the result, bc, is taken a times as an addend; or it means $a\times b\times c$.

28. If $a=1$, $b=2$, and $c=3$, find the value of ab; of bc; of abc.

29. If $a=2$, $b=3$, and $c=4$, find the value of $6a$; of $6ab$; of $6abc$; of bc; of abc.

30. Analyze $-4b$.

Ans. $-4b$ is a term; -4 is the coefficient; b is the base. It means $-b-b-b-b$.

31. If $b=5$, find the value of $-4b$; of $-3b$; of $-ab$.

32. Express as concisely as possible that x is taken 5 times as a subtrahend; that 3 is taken 5 times as a subtrahend.

33. Analyze $0b$.

Ans. $0b$ is a term; 0 is the coefficient, b is the base. It means that b is taken the same number of times both as an addend and as a subtrahend; the result is 0.

34. In $0b$, is it possible to find how many times b is used both as an addend and as a subtrahend?

35. What is meant by the expression, multiply 6 by 0?

TERMS — EXPONENTS

PROPOSITION VI. AGREEMENT

A positive exponent shows how many times the base is used as a multiplier.

Thus, $a^{+3} = a \times a \times a$, where a is used 3 times as a multiplier.

The expression, a^{+3} is a term; $+1$, understood, is the coefficient (p. 12); a is the base; $+3$ is the exponent. A term always consists of these three parts, but one or two parts may be understood.

PROPOSITION VII. THEOREM

A negative exponent shows how many times the base is used as a divisor.

To prove that $$a^{+3} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$$

a^{-3} means that a is used 3 times as a multiplier. Since '$-$' means the opposite of '$+$,' a^{-3} means that a is used 3 times in a sense opposite to a multiplier, or 3 times as a divisor, or $a^{-3} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$.

Hence, the principle, Q.E.D.

PROPOSITION VIII. THEOREM

A zero exponent shows that the base is used the same number of times both as a multiplier and as a divisor; the value of a number with zero exponent is 1.

$$0 = +3 - 3$$

$$a^0 = a^{+3-3} = \frac{a \times a \times a}{a \times a \times a} = 1$$

In the same way, it may be shown that $a^0 = a^{+4-4}$, a^{+5-5}, and so on.

Hence, the principle, Q.E.D.

36. Analyze $3a^4$.

Ans. $3a^4$ is a term; 3 is the coefficient; a is the base; $+4$ is the exponent. It means $a^4 + a^4 + a^4$, or that a is used 4 times as a multiplier, and the result 3 times as an addend. It is read $3a$ to the 4th power, or $3a$ to the 4th.

37. In $-4b$, what is the exponent?

38. In $-b^4$, what is the coefficient?

39. Write a, expressing coefficient and exponent.

40. Analyze $-4a^6$, stating coefficient, base, exponent, and meaning.

41. Write in simplest form: 2 used as a multiplier 5 times; a used as a multiplier 5 times.

42. Write in simplest form: a used as a multiplier 6 times and the result as an addend 4 times; a used as a multiplier 6 times and the result as a subtrahend 4 times.

43. If $b = 5$, find the value of: b^{+4}; b^3; b^5; b^2.

44. If x is 3 and a is 2, find the value of: x^a; $2x^a$; $3x^a$.

45. Analyze $3a^{-4}$.

Ans. $3a^{-4}$ is a term; 3 is the coefficient; a is the base; -4 is the exponent. It means $\frac{1}{a^4} + \frac{1}{a^4} + \frac{1}{a^4}$, or that a is used 4 times as a divisor and the result 3 times as an addend. It is read $3a$ to the minus 4th power, or $3a$ to the minus 4th.

46. Find the value of: 2^{-3}; 3^{-2}; 5^{-2}; 5^{-3}; 5^{-1}.

47. If x is 3 and a is 2, find the value of: x^{-a}; $9x^{-a}$; $27x^{-a}$.

48. Analyze 3^0.

Ans. 3^0 is a term; 3 is the base; 0 is the exponent. It means that 3 is taken the same number of times both as a multiplier and as a divisor. $3^0 = 1$; 3^0 is read *3 to zero power.*

49. In 3^0 is it possible to find how many times 3 is used both as a multiplier and as a divisor?

50. What is meant by the expression, "use 6 0 times as a factor"?

SIGNS OF EQUALITY, INEQUALITY, AND AGGREGATION

If two expressions are equal, the sign of equality, $=$, may be placed between them; the whole is an equation. If two expressions are unequal, the sign of inequality, $>$, with the opening toward the larger, may be placed between them; the whole is an inequality. In each case, the part on either side of the sign is a member and may be made up of terms which are regarded as separated by '$+$' signs understood.

I. $8+6-4=10$ II. $8+6-4>5$ III. $8+6-4<15$

I. is an equation; II. and III. are inequalities; II. is read, $8+6-4$ is greater than 5; III. is read, $8+6-4$ is less than 15. In each case, the left-hand member is composed of the terms $+8$, $+6$, and -4 which may be regarded as connected by '$+$' signs understood.

If two or more quantities are to be subjected to the same operation, or are to be regarded as forming a single quantity, they may be placed within parentheses, ( ); brackets, [ ]; or braces, { }; or under or above a vinculum, $\overline{\qquad}$. Thus,

$+6(x-y)^2$ is a term; $+6$, the coefficient; $x-y$, the base; 2, the exponent; it means that $x-y$ is to be taken 2 times as a multiplier and the result 6 times as an addend.

$(a+b)(x+y)$ is a term; $a+b$, the coefficient, $x+y$, the base; 1, the exponent; it means that $x+y$ is to be taken $a+b$ times as an addend, or that $x+y$ is to be multiplied by $a+b$.

By the use of the proper signs, state that:

51. The sum of 8 and 9 is equal to the sum of $+a$ and $-b$.

52. The sum of 8 and -4 is greater than the sum of 7 and -5.

53. The sum of 8 and 3 is less than the sum of 16 and -2.

54. The sum of a, $-b$, and $-c$ is to be multiplied by 6.

55. The sum of $6a$ and $-7b$ is to be multiplied by $a-b$.

SEQUENCE OF SIGNS

When more than one sign occurs in the same member, an expedient is sometimes needed to prevent a double meaning.

When the sign next after '+' or '−' is '×' or '÷,' it is agreed that the signs of multiplication and division shall be used first. Thus,

$$24 + 6 \times 2 = 36 \qquad 24 - 6 \div 2 = 21$$

The algebraic notation demands this because '+' and '−' signs are regarded as belonging to the numbers before which they stand. $24 + 6 \times 2$, the left-hand member of the first equation, is made up of the terms 24 and $+6 \times 2$; in the same way, $24 - 6 \div 2$ is made up of the terms 24 and $-6 \div 2$.

When the sign next after '÷' is '×' or '÷,' a sign of aggregation must be used to denote the order in which the signs are to be used. Thus,

$$(24 \div 6) \times 2 = 8 \qquad 24 \div (6 \times 2) = 2$$

$$(24 \div 6) \div 2 = 2 \qquad 24 \div (6 \div 2) = 8$$

$24 \div 6 \times 2$ is ambiguous

Find the value of:

56. $10 + 5 \times 6 - 8 \div 2$

57. $12 - 2 \times 3 + 4 \times 5$

58. $16 \div 2 + 8 \div 4 - 6$

59. $(48 \div 6) \times 3 - 3 \times 2$

60. $48 \div (6 \times 2) + 3 \times 2$

61. $(6 \div 2) \div 3 + 5 \times 6$

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the numerical values of:

62. $6(cd - ac) - 2(bd - bc)$

63. $8(abc - d) + (c + bcd)$

64. $bcd - (acd - abd - ad)$

65. $2(abcd - 3abc + 4cd)$

66. $3d^3 - 2a^3 - 3c^2 - 4b^3$

67. $2a^4b^2c^3d - 16(b^2 - a^2)$

68. $3(c^2 - b^2)^2 + 2(a + b + c)^3$

69. $(a^3 + 3a^2b + 3ab^2 + b^3)^2$

70. $(b + a)^2 - (b - a)^2 - (b^2 + ab + a^2) + c^3.$

71. $(b + a)^3 - (b - a)^3 + (b^2 + a^2)(b^2 - a^2) - c^2.$

ADDITION AND SUBTRACTION

DEVELOPMENT

The fact that the whole is equal to the sum of all its parts may be illustrated:

$$9a = 5a + 4a$$

By the omission of each term in succession, three problems arise:

1. What $= 5a + 4a$?
2. $9a =$ what $+ 4a$?, or what $= 9a - 4a$?
3. $9a = 5a +$ what?, or what $= 9a - 5a$?

Problem 1 gives rise to addition. It means what is the whole when the parts are $5a$ and $4a$, or what is the sum of $5a$ and $4a$? Addition is the process of finding the whole when the parts are given. The whole is the sum or amount; the parts to be united are addends. The sign of addition is '+,' read plus.

Problems 2 and 3 give rise to subtraction. The former means what must be added to $4a$ to produce $9a$, or what is the difference between $9a$ and $4a$? Subtraction is the process of finding one of two quantities when the other and their sum are given. The sum is the minuend; the part given is the subtrahend; the part required is the remainder. The sign of subtraction is '−,' read minus.

NOTE. In algebra, $9a - 4a$ is usually regarded as $(+9a) + (-4a)$, or as the sum of $+9a$ and $-4a$.

The use of the signs '+' and '−' to denote opposite directions broadens the notation of addition and subtraction.

To add a number with either sign is to subtract the same number with the opposite sign. To subtract a number with either sign is to add the same number with the opposite sign. If the signs outside the parentheses are used to denote addition and subtraction, these laws may be expressed:

1. $+(+a)=-(-a), \quad +(-a)=-(+a)$
2. $-(+a)=+(-a), \quad -(-a)=+(+a)$

The first law may be illustrated: To add \$ a savings is to subtract \$ a expenses; to add \$ a expenses is to subtract \$ a savings. The second law may be illustrated: To subtract \$ a savings is to add \$ a expenses; to subtract \$ a expenses is to add \$ a savings.

Hence, the rules for addition and subtraction might be stated: To add a number with direction, change its sign and proceed as in subtraction; to subtract a number with direction, change its sign and proceed as in addition. The latter is generally employed for subtraction, but a different rule is substituted for addition.

For convenience, the coefficient of a term may be regarded as any one of its factors or as any combination of them. All of the term except its coefficient may be regarded as its unit.

Thus, the coefficient of $-4a^3b^2c$ is usually regarded as -4 and the unit as a^3b^2c; the term then means that a^3b^2c is taken 4 times as a subtrahend. The coefficient may be regarded as $-4a^3$ and the unit as b^2c; the term then means that b^2c is taken $4a^3$ times as a subtrahend. If a^3 is the coefficient, $-4b^2c$ is the unit; and so on. See p. 29.

In order that terms may be added or subtracted their units must be the same. In this event the terms are said to be similar.

Thus, $3a^2$ may be added to $5a^2$; $3a^3b^2c$ may be subtracted from $5a^3b^2c$, because their units are the same. $3a^2$ cannot be subtracted from $5a^3$, because their units are not the same.

ADDITION — COEFFICIENTS

PROPOSITION IX. THEOREM

To add when the signs are alike, write the sum and use the common sign; to add when the signs are unlike, write the difference and use the sign of the greater.

Sum, difference, and greater are here used without reference to signs.

	$+3$	-3	$+3$	-3
Let us take	$+2$	-2	-2	$+2$
To prove the sums	$+5$	-5	$+1$	-1

$+2$ added to $+3$ means that some unit is taken 3 times as an addend and then 2 times more as an addend. 2 times as an addend increases 3 times as an addend and the result becomes $3+2$, or 5 times as an addend, or $+5$.

-2 added to -3 means that some unit is taken 3 times as a subtrahend and then 2 times more as a subtrahend. 2 times as a subtrahend increases 3 times as a subtrahend and the result becomes $3+2$, or 5 times as a subtrahend, or -5.

-2 added to $+3$ means that some unit is taken 3 times as an addend and then 2 times as a subtrahend. 2 times as a subtrahend cancels 2 times as an addend and the result becomes $3-2$, or 1 time as an addend, or $+1$.

$+2$ added to -3 means that some unit is taken 3 times as a subtrahend and then 2 times as an addend. 2 times as an addend cancels 2 times as a subtrahend and the result becomes $3-2$, or 1 time as a subtrahend, or -1.

Hence, the principle, Q.E.D.

Add:

	-8	6	8	-6	-8	-6	9
1.	3	-4	-3	-7	$+7$	$+4$	3
	6	-6	-9	$+9$	9	-9	7
2.	-8	-5	-8	$+8$	-8	8	-3

ADDITION — TERMS

PROPOSITION X. THEOREM

To add terms having a common unit, retain the common unit and add the coefficients.

	ma
Let us take	$\underline{na}$
To prove the sum	$(m+n)a$

$ma + na$ means that a is taken m times as an addend and then times more, or $m+n$ times; the result is $(m+n)a$.
Hence, the principle, Q.E.D.

Add:

3. $\begin{array}{r} 3a \\ \underline{-4a} \end{array}$ $\quad \begin{array}{r} a \\ \underline{-2a} \end{array}$ $\quad \begin{array}{r} 7x \\ \underline{-8x} \end{array}$ $\quad \begin{array}{r} y \\ \underline{-3y} \end{array}$ $\quad \begin{array}{r} 9x \\ \underline{-4x} \end{array}$ $\quad \begin{array}{r} -4abc \\ \underline{+9abc} \end{array}$

4. $\begin{array}{r} 9x \\ \underline{-3x} \end{array}$ $\quad \begin{array}{r} -9b \\ \underline{6b} \end{array}$ $\quad \begin{array}{r} 7y \\ \underline{-8y} \end{array}$ $\quad \begin{array}{r} 5ab \\ \underline{-6ab} \end{array}$ $\quad \begin{array}{r} -6xy \\ \underline{-8xy} \end{array}$ $\quad \begin{array}{r} -3xyz \\ \underline{-7xyz} \end{array}$

5. $\begin{array}{r} 7y \\ \underline{8y} \end{array}$ $\quad \begin{array}{r} -6c \\ \underline{+5c} \end{array}$ $\quad \begin{array}{r} -4by \\ \underline{-6by} \end{array}$ $\quad \begin{array}{r} 9xy \\ \underline{-9xy} \end{array}$ $\quad \begin{array}{r} 7ab \\ \underline{-6ab} \end{array}$ $\quad \begin{array}{r} -7abcd \\ \underline{+8abcd} \end{array}$

6. $\begin{array}{r} ax \\ \underline{-bx} \end{array}$ $\quad \begin{array}{r} acy \\ \underline{acy} \end{array}$ $\quad \begin{array}{r} acz \\ \underline{-bdz} \end{array}$ $\quad \begin{array}{r} -aby \\ \underline{+aby} \end{array}$ $\quad \begin{array}{r} -abcx \\ \underline{-bcdx} \end{array}$ $\quad \begin{array}{r} abcdy \\ \underline{-abcdy} \end{array}$

7. $\begin{array}{r} -3a^2+4b^2 \\ \underline{6a^2-b^2} \end{array}$ $\quad \begin{array}{r} -7ab+6 \\ \underline{-2ab-4} \end{array}$ $\quad \begin{array}{r} 5a^2b^2-2x \\ \underline{-3a^2b^2+2x} \end{array}$ $\quad \begin{array}{r} 5x^2-xy \\ \underline{-2x^2+xy} \end{array}$

Ex. 3. The common unit is a; the sum of the coefficients, -1; the result, $-a$.

Ex. 7. We begin at the left and add one column at a time.

$(-3a^2)+(+6a^2)=+3a^2$; $(+4b^2)+(-b^2)=+3b^2$. *Ans.* $3a^2+3b^2$

ADDITION — POLYNOMIALS

Add:

8.

$$\begin{array}{r} 3a^2+3ab+4b^2 \\ \underline{3a^2-2ab-4b^2} \end{array}$$

9.

$$\begin{array}{r} x^2-2xy+y^2 \\ \underline{x^2+2xy+y^2} \end{array}$$

10.

$$\begin{array}{r} 2a^2b-3ab^2+b^3 \\ \underline{2a^2b+3ab^2-b^3} \end{array}$$

11.

$$\begin{array}{r} -5x^3-5x^2-x+7 \\ -7x^3+3x^2+8x-2 \\ \underline{3x^3-8x^2-9x+8} \end{array}$$

12.

$$\begin{array}{r} 3x^3-4x^2-x+7 \\ 2x^3-x^2+3x-10 \\ \underline{6x^3-2x^2-4x-5} \end{array}$$

13.

$$\begin{array}{r} a^4+2a^2-5a-3 \\ -3a^4+7a^2-10a+12 \\ \underline{-3a^4+5a^2-2a-13} \end{array}$$

14.

$$\begin{array}{r} 3a^2-6ab+b^2 \\ -7a^2+8ab-9b^2 \\ \underline{-5a^2-8ab+9b^2} \end{array}$$

15.

$$\begin{array}{r} 3x^2-6xy+3y^2 \\ 3x^2+6xy+3y^2 \\ \underline{x^2-xy+y^2} \end{array}$$

16.

$$\begin{array}{r} x^2-2xy+y^2 \\ -3x^2+2xy-y^2 \\ \underline{5x^2+6xy+3y^2} \end{array}$$

17. $4x^4+3a^2x^2+a^4$, $-3x^4-3a^2x^2-5a^4$, $6x^4+5a^2x^2$, $-2a^2x^2-7x^4+3a^4$, x^4-a^4, $-5a^4-3a^2x^2-7a^4$.

18. $a^3-3a^2b+3ab^2-b^3$, $a^3+3a^2b+3ab^2+b^3$, a^3-b^3, a^3+b^3, $4a^2b-6ab^2$, $3a^3-7b^3+6ab^2-7a^2b$.

19. $x^2-2xy+y^2$, $x^2+2xy+y^2$, x^2-y^2, x^2+y^2, $5x^2-10xy+5y^2$, $3x^2+6xy+3y^2$, $4xy$, $7x^2-7y^2$, $8x^2+8y^2$.

20. $a^3+3a^2b-3ab^2+b^3$, $6a^2b+12ab^2$, a^3-b^3, a^3+b^3, $5a^3+11a^2b+6ab^2-7b^3$, $17ab^2$, $7a^3+6a^2b+8b^3$.

Ex. 14. In the first column, the sum of 3 and -7 is -4; of -4 and -5, -9. Or, the sum of the '$-$' coefficients is -12; of the '$+$' coefficients, $+3$; of the results, -9.

NOTE. A polynomial is an algebraic expression of more than one term.

SUBTRACTION — COEFFICIENTS

PROPOSITION XI. THEOREM

To subtract, change the sign of the subtrahend and proceed as in addition.

	$+2$	-2	-2	$+2$
Let us take	$+3$	-3	$+3$	-3
To prove the remainders	-1	$+1$	-5	$+5$

By definition, the remainder is what must be added to the subtrahend to produce the minuend. This will be the sum of what must be added to the subtrahend to produce 0 and of what must be added to 0 to produce the minuend.

The subtrahend with its sign changed will cancel the subtrahend, and the minuend added to 0 will produce the minuend; *i.e.*, the remainder is the sum of the subtrahend with its sign changed and of the minuend.

Hence, the principle, Q.E.D.

Subtract:

	-6	-3	-6	-9	-5	$+6$	-3
21.	$+4$	-8	$+7$	-5	-9	-8	-7
	$+8$	8	-8	6	8	-3	3
22.	-4	-8	8	9	-3	-8	7

PROPOSITION XII. THEOREM

To subtract terms having a common unit, retain the common unit and subtract the coefficients.

	ma
Let us take	na
To prove the remainder	$(m-n)a$

$ma - na$ means that a is taken m times as an addend and then n times less, or $m - n$ times; the result is $(m-n)a$.

Hence, the principle, Q.E.D.

SUBTRACTION — TERMS

Subtract:

	$-4x$	$-a$	$-7y$	b	$-2x$	$4abc$
23.	$+3x$	$+2a$	$+8y$	$3b$	$-7x$	$9abc$
	$-6y$	c	$-5a$	$+8x$	$-2ab$	$5bxy$
24.	$-7y$	$9c$	$+3a$	$-10x$	$-5ab$	$-8bxy$
	$7x$	$-5y$	$-7b$	$+7b$	$-ab$	$7axy$
25.	$-8x$	$+6y$	$-3b$	$+3b$	$+9ab$	$-9axy$

Ex. 23. The common unit is x; the coefficient of the minuend minus the coefficient of the subtrahend, -7; the result, $-7x$.

Subtract:

26.	**27.**	**28.**
$3a^2+6ab-3b^2$	$a^2+2ab+b^2$	$a^2x^2-2abx+b^2$
$4a^2-8ab-4b^2$	$a^2-2ab+b^2$	$-a^2x^2+2abx-b^2$

29. $5a^3+6a^2b+5ab^2+6b^3$ from $a^3+3a^2b+3ab^2+b^3$.

30. $a^2+b^2+c^2+2ab+2ac$ from $ab+ac-c^2+a^2-b^2$.

31. $3a^4-6a^3b+7a^2b^2-8ab^3$ from $2ab^2-3b^4+3a^4+7a^2b^2$.

32. $3a^4-a^2+7a-14$ from $11a^4-2a^3+3a^2-8a$.

33. $abcxy+2aby-4bx$ from $2abcxy-aby+bx-3$.

34. $10c-a-b+5d+6a-15c+3d$ from $25a-b-5c+8d-20a$.

Ex. 26. *Ans.* $-a^2+14ab+b^2$. We begin at the left and subtract one column at a time.

MISCELLANEOUS

Add:

35.

$$\begin{array}{l} ax + a^2y \\ bx + b^2y \\ cx + c^2y \\ \hline \end{array}$$

36.

$$\begin{array}{l} (2a - 3b)x \\ (3a - 2b)x \\ (4a + 6b)x \\ \hline \end{array}$$

37.

$$\begin{array}{l} (3a - b + 3c)x \\ (3a + 2b + 3c)x \\ (4a - 2b + 2c)x \\ \hline \end{array}$$

Subtract:

38.

$$\begin{array}{l} ax - by \\ bx - cy \\ \hline \end{array}$$

39.

$$\begin{array}{l} (2a - 3b)x \\ (3a + 2b)x \\ \hline \end{array}$$

40.

$$\begin{array}{l} (4a - 2b + c)x \\ (3a + 3b - 2c)x \\ \hline \end{array}$$

41. From the sum of $2x^3 - x^2y - 5xy^2$ and $3x^2y - 5xy^2 - 4y^3$ take the sum of $-2x^3 - 7x^2y - 6y^3$ and $-6xy^2 + 5y^3$.

42. From the sum of $a^4 - 1$, $3a^4 - 5$, $a^3 - 6a^2$, and $2a^3 - 10a^2 - 7a$ subtract the sum of $-3a^4 + 2a^2 - 5a$ and $-5a^3 - 12a^2 + 3$.

43. From $16a^2b^2 - 12bc - 14b^2$ subtract the sum of $a^2b^2 - 3bc + 4$, $-2a^2b^2 + 5bc - 6$, $5a^2b^2 - 6bc + 8$, $3a^2b^2 + 2bc - 7$, and $10a^2b^2 + 8bc + 5b^2$.

44. From the sum of $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $x^4 + 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $x^3y - 3x^2y^2 + 3xy^3 - y^4$, and $x^4 + 3x^3y + 3x^2y^2 + xy^3$ subtract the sum of $x^4 + 2x^2y^2 + y^4$, $x^4 - 2x^2y^2 + y^4$, $x^4 + x^2y^2 + y^4$, $x^4 - x^2y^2 + y^4$, $3x^3y - 3xy^3$, and $-8x^3y - 9xy^3$.

Ex. 35. In the first column, the common unit is x; the sum of the coefficients, $a + b + c$; the result, $(a + b + c)x$.

The result in the second column is $(a^2 + b^2 + c^2)y$. *Ans.* $(a + b + c)x + (a^2 + b^2 + c^2)y$.

Ex. 36. The common unit is x; the sum of the coefficients, $9a + b$; the result, $(9a + b)x$.

SOLUTION OF PROBLEMS

The representation of numbers by letters not only satisfies the need of a briefer notation in the solution of problems but also simplifies the form of explanation.

1. If a number is increased by 5, the sum is 9. Find the number.

If the number increased by 5 equals 9, the number must be 9 diminished by 5, or 4. By the representation of *the number* by x, this becomes, If $x + 5 = 9$, $x = 9 - 5$, or 4.

2. If a number is diminished by 6, the remainder is 3. Find the number.

If the number diminished by 6 equals 3, the number must be 3 increased by 6, or 9. By the representation of *the number* by x, this becomes, If $x - 6 = 3$, $x = 3 + 6$, or 9.

From these explanations, it appears that a term may be transposed to the other side of an equation by changing its sign. This law renders a simpler explanation possible. Thus,

Ex. 1.

Let	$x =$ the number
then	$x + 5 = 9$
transposing,	$x = 9 - 5$
uniting,	$x = 4$

Ex. 2.

Let	$x =$ the number
then	$x - 6 = 3$
transposing,	$x = 3 + 6$
uniting,	$x = 9$

In the explanation of a problem after it has been solved, the following directions should be observed:

1. State the problem without looking at the book.
2. After every statement give a reason unless the reason adds nothing to clearness.
3. After the equation is formed, do not explain the solution, but declare the result.
4. In the proof, state the first condition and show how it is met; state the second condition and show how it is met; and so on.

3. After purchasing 75 sheep a farmer had 243. How many sheep did he have at first?

$$
\begin{aligned}
\text{Let } x &= \text{number sheep at first, } 168 \\
x + 75 &= \text{number after the purchase} \\
243 &= \text{number after the purchase} \qquad \textsc{Proof.}\quad 168 + 75 = 243 \\
\therefore\ x + 75 &= 243 \\
x &= 243 - 75 = 168
\end{aligned}
$$

Explanation. Let x equal the number of sheep at first. $x + 75$ must equal the number after the purchase; 243 equals the number after the purchase; therefore, $x + 75 = 243$, and $x = 168$, the number of sheep at first.

Proof. The condition is that he was to have 243 sheep after purchasing 75 sheep; 168 sheep + 75 sheep is 243 sheep.

Note. '$\therefore$' is the sign for *therefore*. The answer, 168, is not placed on the first line until the solution is finished.

4. A and B together have \$105 and A has \$75. How much has B? Let $x =$ B's money in dollars.

5. A and B together have \$105 and B has \$30. How much has A? Let $x =$ A's money in dollars.

6. A invests \$2100. How much must he gain to have \$3600?

7. In a pasture containing cows and sheep there are 23 sheep and in all 34 animals. How many cows are there?

8. If A had \$365 more he would have \$570. How much money has he?

9. If a farmer buys 200 sheep and then sells 75 he will have 235 left. How many sheep did he have at first?

10. James has 9 marbles less than Joseph; Joseph has 7 more than Henry; Henry has 27. How many marbles has James? Let $x =$ the number James has.

11. A man saved \$200 more the second year than the first; \$300 less the third year than the second; and saved the third year \$500. How much did he save the first year? Let $x =$ the number of dollars he saved the first year.

MULTIPLICATION AND DIVISION

DEVELOPMENT

When the addends are equal, the fact that the whole is equal to the sum of all its parts, may be illustrated:

$$8a = 2a + 2a + 2a + 2a, \text{ or } 8a = 2a \times 4$$

By the omission of each term in succession, three problems arise:

1. What $= 2a \times 4$?
2. $8a = 2a \times$ what?, or what $= 8a \div 2a$?
3. $8a =$ what $\times 4$?, or what $= 8a \div 4$?

Problem 1 gives rise to multiplication. It means, what is the sum when $2a$ is used 4 times as an addend? or what is 4 times $2a$? Multiplication is the process of finding the sum of equal quantities from one of them and from the number of times it occurs. The sum is the product; the equal addend, the multiplicand; the number of times, the multiplier; either multiplicand or multiplier, a factor.

Multiplication may be expressed by the sign '$\times$' before the multiplier, by the sign '$+$' over the multiplier and the sign '$\times$' before it, by the sign '$-$' over the multiplier and the sign '$\div$' before it, or by juxtaposition when a letter follows another letter or an Arabic numeral. Continued multiplication may be expressed by periods; continued multiplication of the natural series of numbers 1, 2, 3, 4, ⋯, by writing the largest number of the series in the symbol $\underline{|}$. Thus,

$$6¢ \times 2 = 6¢ \times 2^{+1} \qquad a \times b \times c = abc$$
$$6¢ \times 2 = 6¢ \div 2^{-1} \qquad 4 \times (a + b) = 4(a + b)$$
$$4 \times a = 4a \qquad 6 \times 8 \times 5 = 6 \cdot 8 \cdot 5$$
$$4 \times a \times b = 4ab \qquad 1 \times 2 \times 3 = \underline{|3}$$

In a term, every factor or every factor except one denotes times, or is a multiplier. Thus,

abc means that some unit is used c times as an addend, that the result is used b times as an addend, and so on. \$ *abc* means that \$1 is used c times as an addend, that the result is used b times as an addend, and so on.

Each of the other problems gives rise to division and suggests both a common and a distinctive definition. By the definition common to Probs. 2 and 3, division is the process of finding one of two numbers when their product and the other number are given. The product is the dividend; the number given, the divisor; the number required, the quotient.

Problem 2 means, how many times must $2a$ be used as an addend to produce $8a$? or how many times does $8a$ contain $2a$? One case in division is the process of finding how many times one quantity contains another of the same kind. It is expressed by the sign '$\div$,' by the sign ':,' or by the dividend above and the divisor below a bar. Thus,

$$6¢ \div 2¢ = 6¢ : 2¢ \qquad 6¢ \div 2¢ = \frac{6¢}{2¢}$$

Problem 3 means, what must be taken 4 times as an addend to produce $8a$? or what is one part when $8a$ is divided into 4 equal parts? One case in division is the process of separating a number into equal parts. It is expressed by the sign '$\div$' before the divisor, by the sign '$+$' above the divisor and the sign '$\div$' before it, by the sign '$-$' above the divisor and the sign '$\times$' before it, or by the dividend above and the divisor below a bar. Thus,

$$6¢ \div 2 = 6¢ \div 2^{+1} \qquad 6¢ \div 2 = \frac{6¢}{2}$$
$$6¢ \div 2 = 6¢ \times 2^{-1}$$

MULTIPLICATION — SIGNS

PROPOSITION XIII. THEOREM

The product of like signs is '+'; the product of unlike signs is '−.'

	$+3$	-3	$+3$	-3
Let us take	$+2$	-2	-2	$+2$
To prove the products	$+6$	$+6$	-6	-6

$+3 \times +2$ means that $+3$ is taken 2 times as an addend, or $+3+3$, or $+6$.

-3×-2 means that -3 is taken 2 times as a subtrahend, or $-(-3)-(-3)$, or $+3+3$, or $+6$.

$+3 \times -2$ means that $+3$ is taken 2 times as a subtrahend, or $-(+3)-(+3)$, or $-3-3$, or -6.

$-3 \times +2$ means that -3 is taken 2 times as an addend, or $+(-3)+(-3)$, or $-3-3$, or -6.

Hence, the principle, Q.E.D.

1. Declare the product: -4×-8; $-4 \times +8$; 4×-8; 4×8; 3×4; $+3 \times +4$; $-3 \times +4$; -3×-4; $+3 \times -4$.

PROPOSITION XIV. THEOREM

There must be three cases in multiplication: when the bases are the same, when the exponents are the same, and when neither bases nor exponents are the same.

To multiply we must have two terms, a multiplier and a multiplicand. If we have two terms, the bases must be the same, as $4^2 \times 4^3$; or the exponents the same, as $4^3 \times 5^3$; or neither bases nor exponents the same, as $2^3 \times 4^2$.

Hence, the principle, Q.E.D.

NOTE. $4^3 \times 4^3$ may be regarded as the case when the bases are the same, or the case when the exponents are the same.

MULTIPLICATION — BASES THE SAME

Proposition XV. Theorem *

To multiply when the bases are the same, write the common base, and over it, the exponent of the multiplicand plus the exponent of the multiplier.

Let us take $4^2 \times 4^3$.

$$\begin{aligned} 4^2 &= 4 \times 4 \\ 4^3 &= 4 \times 4 \times 4 \\ \hline 4^2 \times 4^3 &= 4 \times 4 \times 4 \times 4 \times 4 \\ &= 4^5 \end{aligned}$$

Analyzing the product, we see that the base, 4, is the common base; that the exponent, 5, is the exponent of the multiplicand plus the exponent of the multiplier.

Hence, the principle, Q.E.D.

Simplify:

2. $2^2 \times 2^3$; 2×2^4
3. $6^3 \times 6^4$; $7^2 \times 7^4$
4. $x^2 \times x^3$; $x^4 \times x^6$
5. $y^3 \times y^4$; $y^2 \times y$
6. $x^4 \times x^5$; $y^3 \times y^6$
7. $a^m \times a^n$; $x^p \times x^q$
8. $(a+b)^2 \times (a+b)^3$
9. $(x-y)^3 \times (x-y)$

Ex. 8. The common base is $a+b$; the product, $(a+b)^5$.

Simplify:

10. $-3a^2b \times 2ab$
11. $-4xy^2 \times -5x^2y^3$
12. $+3a^2b^2 \times -6ab^2$
13. $-4abc \times -6a^2b$
14. $-3a^2bc^2 \times 8a^3bc^2$
15. $-6a^3b^3c^2 \times ab^2c^3$

Ex. 10. $-3 \times +2 = -6$; $a^2 \times a = a^3$; $b \times b = b^2$; the entire product is $-6a^3b^2$.

* For negative and fractional exponents, see p. 167.

MULTIPLICATION — BASES THE SAME

Multiply:

16. $a^3 - 3a^2b + 3ab^2 - b^3$ by a

17. $a^4 - 4a^3b^3 - 6a^2b^2$ by $-a^2b$

18. $6a^2b - 4ab^3 - b^4$ by ab^2

19. $a^2 + 2ab + b^2$ by $-ab$

20. $5x^4y - 10x^3y^2$ by $4x^3y$

21. $6x^5y - 15x^4y^2$ by $-5x^2y^3$

22. $3x^2y - 3xy^2$ by $3xy$

23. $2x^2y^2 - 3xy^2$ by $-xy$

Ex. 16.

$$\begin{array}{r} a^3 - 3a^2b + 3ab^2 - b^3 \\ a \\ \hline a^4 - 3a^3b + 3a^2b^2 - ab^3 \end{array}$$

We begin at the left and multiply each term in succession.

$a^3 \times a = a^4$; $-3a^2 \times a = -3a^3$; $3ab^2 \times a = 3a^2b^2$; $-b^3 \times a = -ab^3$.

Multiply:

24. $x + y$ by $x - y$

25. $x + y$ by $x + y$

26. $x^2 + y^2$ by $x^2 - y^2$

27. $x^3 + y^3$ by $x^3 - y^3$

28. $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$

29. $x^2 + xy + y^2$ by $x^2 - xy + y^2$

30. $x^2 - 3xy + y^2$ by $x^2 + 3xy + y^2$

31. $x^4 - x^2y^2 + y^4$ by $x^4 + x^2y^2 + y^4$

32. $3x^2y - 2xy^2$ by $2x^2y - 5xy^2$

33. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 - 2xy + y^2$

Ex. 32.

$$\begin{array}{lll} 3x^2y - 2xy^2 & & \\ 2x^2y - 5xy^2 & & \\ \hline 6x^4y^2 - & 4x^3y^3 & \\ & -15x^3y^3 & + 10x^2y^4 \\ \hline 6x^4y^2 - & 19x^3y^3 & + 10x^2y^4 \end{array}$$

We arrange the terms according to the descending powers of x. We begin at the left, multiply by each term of the multiplier in succession, and add the several products.

MULTIPLICATION — OTHER CASES

PROPOSITION XVI. THEOREM

To multiply when the exponents are the same, write the product of the bases, and over it the common exponent.

Let us take $4^3 \times 5^3$.

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 \\ 5^3 &= 5 \times 5 \times 5 \\ \hline 4^3 \times 5^3 &= (4 \times 5)(4 \times 5)(4 \times 5) \\ &= 20 \times 20 \times 20 \\ &= 20^3 \end{aligned}$$

Analyzing the product, we see that the base, 20, is the product of the bases; that the exponent, 3, is the common exponent.

Hence, the principle, Q.E.D.

Simplify:

34. $2^4 \times 3^4$; $2^x \times 3^x$

35. $3^4 \times 4^4$; $5^a \times 6^a$

36. $(x + y)^3 \times (x - y)^3$

37. $(a^2 + b^2)^2 \times (a^2 - b^2)^2$

PROPOSITION XVII. THEOREM

To multiply when neither bases nor exponents are the same, fulfill one of these conditions.

Let us take $2^3 \times 4^2$.

$$\begin{aligned} 2^3 &= 8^1 \\ 4^2 &= 16^1 \\ \hline 2^3 \times 4^2 &= 8^1 \times 16^1 \end{aligned}$$

Here, the exponents are the same and we can multiply.

$$\begin{aligned} 2^3 &= 2^3 \\ 4^2 &= 2^4 \\ \hline 2^3 \times 4^2 &= 2^3 \times 2^4 \end{aligned}$$

Here, the bases are the same and we can multiply.

Hence, the principle, Q.E.D.

Simplify:

38. $2^3 \times 3^2$; $3^4 \times 2$

39. $5^2 \times 2^3$; $4^3 \times 6^2$

40. $2^3 \times 4^2$; 2×4^2

41. $3^4 \times 9^2$; 3×9^2

MULTIPLICATION

To remove signs of aggregation, it is necessary only to perform the indicated operations.

In every case, the pupil should determine all of the operations that are required before he performs any of them.

Simplify:

42. $6(3m-n)-8(m+n)-(2m-3n)$

43. $8(m+n)-4(m-n)-6(3m-2n)+7(m+2n)$

44. $5(a-b+c)-(2a-3b-2c)-2(a-3b+2c)$

45. $6(2m-n)(3m-2n)-(m-n)(m-3n)$

46. $(a+b)(b+c)-(c+d)(a+d)-(a+c)(b-d)$

47. $(x+1)(x-1)(x^2+1)-(x+1)^2(x-1)^2-(2x^2-2)$

48. $(a-3b-2c)-6(2a+b-c)-(a-b-c)$

49. $(a+b)^3-(a-b)^3-3ab(a-b)+3ab(a+b)$

Ex. 42. This means that $3m-n$ is to be multiplied by 6; $m+n$ by -8; $2m-3n$ by -1; and that the results are to be united. Each term may be written in a vertical column as soon as the multiplication is performed. Thus,

$$\begin{array}{r}
18m - 6n \\
-8m - 8n \\
\underline{-2m + 3n} \\
8m - 11n
\end{array}$$

6 times $3m$ is $18m$; 6 times $-n$ is $-6n$; -8 times m is $-8m$; -8 times n is $-8n$; -1 times $2m$ is $-2m$; -1 times $-3n$ is $+3n$.

Ex. 45. This means that the continued product of $3m-2n$, $2m-n$ and 6; and the continued product of $m-3n$, $m-n$ and -1; are to be united. It is read, 6 times the expression $2m-n$ times the expression $3m-2n$, minus the expression $m-n$ times the expression $m-3n$.

The work may be performed without rewriting the factors, and the terms of the products may be arranged as in the explanation of Ex. 42. By this method, $3m-2n$ must be multiplied by $2m\times 6$, or $12m$, and then by $-n\times 6$, or $-6n$; $m-3n$ must be multiplied by $m\times -1$, or $-m$, and then by $-n\times -1$, or n.

One sign of aggregation may occur within another; to remove them it is necessary only to perform the indicated operations.

Simplify:

50. $3x-\{2[5x-(-4x-\overline{x+y})]-(-x-3y)\}$

51. $6x-[2(3x-2y)-(2x-\overline{3y-4x})-2x]$

52. $x^2-\{[-(x^2-x)-3(x^2+x)-2x]-3(2x^2-x)\}$

53. $-[-2x-\{-(2x-3y)-\overline{3x-2y}\}+2x]$

54. $-[(a+b)^2-(a-b)^2]+[(a+b)^3+(a-b)^3]+4ab$

55. $-3[(a^2+2ab+b^2)(a^2-2ab+b^2)-(a^2-b^2)^2]$

Ex. 50.

$$3x-\{2[5x-(-4x-\overline{x+y})]-(-x-3y)\} \qquad (1)$$

$$3x-2[5x-(-4x-\overline{x+y})]+(-x-3y) \qquad (2)$$

$$3x-10x+2(-4x-\overline{x+y})-x-3y \qquad (3)$$

$$3x-10x-8x-2\times\overline{x+y}-x-3y \qquad (4)$$

$$3x-10x-8x-2x-2y-x-3y \qquad (5)$$

$$-18x-5y \qquad (6)$$

(1) The two terms within { }, viz., $+2[5x-(-4x-\overline{x+y})]$, and $-(-x-3y)$, must be multiplied by -1; this gives (2).

(2) The two terms within [ ], viz., $+5x$, and $-(-4x-\overline{x+y})$, must be multiplied by -2; the two terms within ( ), viz., $-x$, and $-3y$, must be multiplied by $+1$; this gives (3).

(3) The two terms within ( ), viz., $-4x$, and $-\overline{x+y}$, must be multiplied by $+2$; this gives (4).

(4) The two terms under $\overline{\quad\quad}$, viz., $+x$, and $+y$, must be multiplied by -2; this gives (5).

(5) The terms in (5) must be united; this gives (6), the result in its simplest form.

NOTE. In reading the explanation of a complex process, the pupil will find it of advantage to perform each part on a piece of paper and to compare his result before proceeding to the next. Each step by itself is simple; the seeming complexity arises from viewing all the results at once.

DIVISION — SIGNS

Proposition XVIII. Theorem

The quotient of like signs is '$+$'; the quotient of unlike signs is '$-$.'

Let us take	$+\div+,\ -\div-,\ +\div-,\ -\div+$
To prove the quotients	$+\ ,\ +\ ,\ -\ ,\ -$

By definition, the quotient is what must be multiplied by the divisor to produce the dividend.

In $+\div+$, to produce the dividend, the divisor must be multiplied by $+$. Therefore, $+\div+=+$.

In $-\div-$, to produce the dividend, the divisor must be multiplied by $+$. Therefore, $-\div-=+$.

In $+\div-$, to produce the dividend, the divisor must be multiplied by $-$. Therefore, $+\div-=-$.

In $-\div+$, to produce the dividend, the divisor must be multiplied by $-$. Therefore, $-\div+=-$.

Hence, the principle, Q.E.D.

1. Declare the quotient: $32\div-4$; $-32\div-4$; $-32\div4$; $32\div4$; $12\div3$; $+12\div+3$; $-12\div-3$; $+12\div-3$; $-12\div+3$.

Proposition XIX. Theorem

There must be three cases in division: when the bases are the same, when the exponents are the same, and when neither bases nor exponents are the same.

To divide we must have two terms, a dividend and a divisor. If we have two terms, the bases must be the same, as $4^5\div4^2$; or the exponents the same, as $20^3\div4^3$; or neither bases nor exponents the same, as $2^6\div4^2$.

Hence, the principle, Q.E.D.

Note. $4^3\div4^3$ may be regarded as the case when the bases are the same, or the case when the exponents are the same.

DIVISION — BASES THE SAME

PROPOSITION XX. THEOREM *

To divide when the bases are the same, write the common base and over it the exponent of the dividend minus the exponent of the divisor.

Let us take $4^5 \div 4^2$.

$$\frac{4^5}{4^2} = \frac{4 \times 4 \times 4 \times 4 \times 4}{4 \times 4}$$
$$= 4 \times 4 \times 4$$
$$= 4^3$$

Analyzing the quotient, we see that the base, 4, is the common base; that the exponent, 3, is the exponent of the dividend minus the exponent of the divisor.

Hence, the principle, Q.E.D.

Simplify:

2. $2^5 \div 2^2$; $2^5 \div 2^4$
3. $6^7 \div 6^4$; $7^6 \div 7^2$
4. $x^5 \div x^2$; $x^{10} \div x^4$
5. $y^7 \div y^3$; $y^2 \div y$
6. $x^9 \div x^4$; $y^9 \div y^3$
7. $a^{m+n} \div a^n$; $x^{p+q} \div x^p$
8. $(a+b)^5 \div (a+b)^2$
9. $(x-y)^4 \div (x-y)$

Ex. 8. The common base is $a+b$; the quotient, $(a+b)^3$.

Simplify:

10. $-6a^3b^2 \div -3a^2b$
11. $20x^3y^5 \div -4xy^2$
12. $-18a^3b^4 \div 3a^2b^2$
13. $24a^3b^2c \div -6a^2b$
14. $-24a^5b^2c^4 \div 8a^3bc^2$
15. $-6a^4b^5c^5 \div -6a^3b^3c^3$

Ex. 10. $-6 \div -3 = 2$; $a^3 \div a^2 = a$; $b^2 \div b = b$; the entire quotient is $2ab$.

* For negative and fractional exponents, see p. 169.

[illegible]

$$- 15\, x^4y^3 - 10\, x^3y^4$$

$$- 15\, x^4y^3 - 10\, x^3y^4$$

[illegible] the terms according to the descending powers of x. We [illegible] term of the dividend by the first term of the divisor, [illegible] multiply the whole divisor by the quotient. ($3x^2y$ – [illegible] $- 4x^3y^3$; subtract the result from the dividend, and so [illegible].

[illegible] In subtracting each partial product, care should be taken to use [illegible] in order of the descending powers of x. See p. 41.

DIVISION—EXPONENTS THE SAME

PROPOSITION XXI. THEOREM

To divide when the exponents are the same, write the quotient of the bases and over it the common exponent.

Let us take $20^3 \div 4^3$.

$$\frac{20^3}{4^3} = \frac{20 \times 20 \times 20}{4 \times 4 \times 4}$$
$$= 5 \times 5 \times 5$$
$$= 5^3$$

Analyzing the quotient, we see that the base, 5, is the quotient of the bases; and that the exponent, 3, is the common exponent.

Hence, the principle, Q.E.D.

Simplify:

34. $6^4 \div 3^4$; $6^x \div 2^x$

35. $12^4 \div 4^4$; $30^a \div 6^a$

36. $(x^2 - y^2)^3 \div (x - y)^3$

37. $(a^4 - b^4)^2 \div (a^2 + b^2)^2$

PROPOSITION XXII. THEOREM

To divide when neither bases nor exponents are the same, fulfill one of these conditions.

Let us take $4^2 \div 2^3$.

$$\frac{4^2}{2^3} = \frac{16^1}{8^1}$$

Here the exponents are the same and we can divide.

$$\frac{4^2}{2^3} = \frac{2^4}{2^3}$$

Here the bases are the same and we can divide.

Hence, the principle, Q.E.D.

Simplify:

38. $6^3 \div 2^2$; $18^2 \div 9$

39. $10^4 \div 2^3$; $36^2 \div 6$

40. $2^8 \div 4^2$; $4^4 \div 2$

41. $3^5 \div 9^2$; $9^2 \div 3$

DIVISION

It is sometimes necessary to diminish the number of terms in an expression without changing its value. This is done by dividing the terms by common factors and introducing signs of aggregation.

Express as one term; first, with a common '+' factor and second with a common '−' factor:

42. $5a - 5b$

43. $-6x + 6y$

44. $ab - ac$

45. $-ab + ac$

46. $-3a^2b + 3ab$

47. $3b^2c - 3bc^2$

48. $-ax - bx + x$

49. $ab + a^2b^2 - ab^3$

Ex. 42. $+5$ is a common factor; 5 is contained in $5a$, a times, 5 is contained in $-5b$, $-b$ times; $5a - 5b = 5(a - b)$.

-5 is a common factor; -5 is contained in $5a$, $-a$ times; -5 is contained in $-5b$, b times; $5a - 5b = -5(-a + b)$.

Express as two terms; of the four possible forms, select one that has the same unit in both terms:

50. $ax - ay + bx - by$

51. $bx - by - cx + cy$

52. $bx + by + cx + cy$

53. $ax^2 + by^2 + ay^2 + bx^2$

54. $ax^2 + by^2 - ay^2 - bx^2$

55. $ax^2 - bx^2 - by^2 + ay^2$

Ex. 50. The four possible forms are: $a(x - y) + b(x - y)$; $-a(-x + y) - b(-x + y)$; $a(x - y) - b(-x + y)$; and $-a(-x + y) + b(x - y)$. Either the first or the second answers the requirement; the first is preferable.

Express as one term:

56. $a(x - y) + b(x - y)$

57. $b(x - y) - c(x - y)$

58. $b(x + y) + c(x + y)$

59. $a(x^2 + y^2) + b(x^2 + y^2)$

60. $a(x^2 - y^2) - b(x^2 - y^2)$

61. $a(x^2 + y^2) - b(x^2 + y^2)$

Ex. 56. The common factor is $x - y$; $x - y$ is contained in $a(x - y)$, a times; in $b(x - y)$, b times; $a(x - y) + b(x - y) = (x - y)(a + b)$.

The terms in both dividend and divisor must be arranged according to the descending powers of some letter. It is interesting to note, however, that if this law is disregarded for any time and then strictly observed, the right quotient may still be found. Thus,

$$(x^4 + x^2y^2 + y^4) \div (x^2 + xy + y^2) = x^2 - xy + y^2$$

We will arrange the terms incorrectly, find two terms of the quotient, rearrange the divisor and the last remainder correctly, and observe the law strictly to the end.

$$
\begin{array}{l|l}
xy + x^2 + y^2 & x^2y^2 + x^4 + y^4 \quad (xy - x^2 \;\vdots\; + 2x^2 - 2xy + y^2 \\
\hdashline
x^2 + xy + y^2 & \underline{x^2y^2 + x^3y + xy^3} \\
 & \qquad - x^3y - xy^3 + x^4 + y^4 \\
 & \qquad \underline{- x^3y - x^4 - x^2y^2} \\
 & \qquad\qquad x^2y^2 - xy^3 + 2x^4 + y^4 \\
 & \qquad\qquad 2x^4 + x^2y^2 - xy^3 + y^4 \\
 & \qquad\qquad \underline{2x^4 + 2x^3y + 2x^2y^2} \\
 & \qquad\qquad\qquad - 2x^3y - x^2y^2 - xy^3 \\
 & \qquad\qquad\qquad \underline{- 2x^3y - 2x^2y^2 - 2xy^3} \\
 & \qquad\qquad\qquad\qquad\quad x^2y^2 + xy^3 + y^4 \\
 & \qquad\qquad\qquad\qquad\quad \underline{x^2y^2 + xy^3 + y^4}
\end{array}
$$

Although the quotient reduces to $x^2 - xy + y^2$, yet much unnecessary labor has resulted.

Divide:

62. $a^3 + a^2b + a^2c - abc - b^2c - bc^2$ by $a^2 - bc$

63. $3a^4 - 10a^3b + 22a^2b^2 - 22ab^3 + 15b^4$ by $a^2 - 2ab + 3b^2$

64. $6a^4 - a^3b + 2a^2b^2 + 13ab^3 + 4b^4$ by $2a^2 - 3ab + 4b^2$

Divide the product of:

65. $a^2 + ax + x^2$ and $a^3 + x^3$ by $a^4 + a^2x^2 + x^4$.

66. $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$ and $x^2 + 2ax + a^2$ by $x^4 - 2ax^3 + 2a^3x - a^4$.

67. $x^3 + 3x^2y + 3xy^2 + y^3$ and $x^3 - 3x^2y + 3xy^2 - y^3$ by the product of $x^2 + 2xy + y^2$ and $x - y$.

SOLUTION OF PROBLEMS

The form of explanation in the solution of problems involving multiplication and division may also be simplified by the notation of algebra.

1. If a number is multiplied by 5, the result is 20. Find the number.

If the number multiplied by 5 equals 20, the number must be $20 \div 5$, or 4. By the representation of *the number* by x, this becomes, If $x \times 5 = 20$, $x = 20 \div 5$, or 4.

2. If a number is divided by 2, the result is 12. Find the number.

If the number divided by 2 equals 12, the number must be 12 multiplied by 2, or 24. By the representation of *the number* by x, this becomes, If $x \div 2 = 12$, $x = 12 \times 2$, or 24.

The change from '$x \times 5 = 20$' to '$x = 20 \div 5$' may be accomplished by reasoning as above, or by dividing both members of the equation by 5. The change from '$x \div 2 = 12$' to '$x = 12 \times 2$' may be accomplished by reasoning as above, or by multiplying both members of the equation by 2. These second methods of making the changes render a simpler explanation possible.

	Ex. 1		**Ex. 2**
Let	$x =$ the number	Let	$x =$ the number
then	$5x = 20$	then	$x \div 2 = 12$
dividing by 5,	$x = 4$	multiplying by 2,	$x = 24$

In the direct or algebraic solution of problems, the required terms are represented by the letters $x, y, z, \cdots$, and various operations are performed upon them just as if they were known numbers. In the complete solution there are two steps: first, to form equations by placing equal to each other expressions equal to the same thing; and second, to find the values of the unknown quantities in the equations.

NOTE. The pupil should read p. 26 again.

3. If 9 colts cost $225, how much will 1 colt cost?

Let $x =$ cost in dollars of 1 colt, 25

then $9x =$ cost in dollars of all

$225 =$ cost in dollars of all

$\therefore\ 9x = 225$

$x = 25$

PROOF

$9 \times \$25 = \$225$

EXPLANATION. Let $x =$ the cost of 1 colt in dollars; then $9x$ must equal the cost of 9 colts; 225 equals the cost of all in dollars; therefore, $9x = 225$, and $x = 25$, the cost of 1 colt in dollars.

PROOF. The condition is that 9 colts shall cost $225. At $25 each, 9 colts will cost $225.

4. Solve Ex. 3 indirectly or by analysis, and study the difference between the two solutions.

If 9 colts cost $225, 1 colt will cost $\frac{1}{9}$ of $225, or $25.

5. At $25 each how many colts can be bought for $225? Let $x =$ number of colts.

6. If a number is divided by 63, the quotient is 567. Find the number. Let $x =$ the number.

7. Solve Ex. 6 indirectly or by analysis.

8. A boy on being asked how old he was, replied, "If you add to my age in years 3 times my age, and 5 times my age, and subtract twice my age, the result will be 49 years." How old was he?

9. In a company of 60 persons there are 3 times as many women as men. How many women are there? Let $x =$ number of men.

10. In a certain pasture containing horses, oxen, and cows, there are 3 times as many horses as oxen, twice as many cows as oxen, and 120 animals in all. How many horses are there? Let $x =$ the number of oxen.

11. A man has 4 times as many dollar pieces as he has dimes, and altogether he has $28.70. How many pieces of each kind has he? Let $x =$ number of dimes.

SOLUTION OF PROBLEMS

12. The sum of two numbers is 80; if the larger is increased by 8 and the smaller is diminished by 3, 3 times the first result is 14 times the second. Find the numbers.

Let $x =$ the larger, 162

then $80 - x =$ the smaller, 18

$x + 8 =$ larger increased by 8

$77 - x =$ smaller diminished by 3

$3(x + 8) = 14(77 - x)$

$x = 62$

PROOF

1. $62 + 18 = 80$
2. $3(62 + 8) = 14(18 - 3)$

EXPLANATION. Let $x =$ the larger; then $80 - x$ must equal the smaller because their sum is 80; $x + 8$ equals the larger increased by 8, and $77 - x$ equals the smaller diminished by 3; $3(x + 8) = 14(77 - x)$ because 3 times the first result is 14 times the second; whence $x = 62$, the larger, and $80 - x = 18$, the smaller.

PROOF. The first condition is that the sum of two numbers is 80; $62 + 18 = 80$. The second condition is that if the larger is increased by 8 and the smaller is diminished by 3, 3 times the first result is 14 times the second; $3(62 + 8) = 14(18 - 3)$.

13. A has 80 marbles and B has 15. How many must A give to B in order that he may have 4 times as many as B?

14. A pole is divided into three parts; the second part is 3 times as long as the first, and the third part is 6 feet longer than the first; the length of the pole is equal to the excess of 60 feet over the length of the smallest part. What are the lengths of the parts, and the length of the pole?

15. If the length of a square field is increased 20 rods and its width by 30 rods, the area will be increased 2200 square rods. What are its dimensions?

SUGGESTION. The area of a rectangle is equal to the product of its base and altitude. Let $x =$ the length of one side of the square in rods.

16. Find four consecutive integers whose sum is 94.

Let $\qquad x = \text{the first, } 22$

then $\qquad x + 1 = \text{the second, } 23$

$x + 2 = \text{the third, } 24$

$x + 3 = \text{the fourth, } 25$

$4x + 6 = \text{the sum}$

$94 = \text{the sum}$

$\therefore 4x + 6 = 94$

$4x = 94 - 6 = 88$

$x = 22$

PROOF

1. 22, 23, 24, 25, are consecutive integers.
2. Their sum is 94.

17. Solve Ex. 16 indirectly. The superiority of the direct method becomes apparent.

18. The sum of five numbers each of which is greater than the preceding by 4 is 165. Find the numbers.

19. The first of four numbers is 6 less than the second, the second is 8 more than the third, and the fourth is equal to the sum of the other three; their sum is 26. Find the numbers.

20. Find three consecutive numbers whose sum is 102.

21. The sum of three numbers is 75; the second is 5 greater than the first, and the third is 5 greater than the second. What are the numbers?

22. One number exceeds another by 5, and their sum is 29. Find the numbers.

23. A saved $105 in 5 weeks; each week he saved $2 more than during the preceding week. How much did he save the second week?

24. A student decides to read a volume of 450 pages in 3 days and to read 50 pages less each day than on the preceding day. How many should he read the first day?

25. Find the first and last of five consecutive numbers whose sum is 150.

SOLUTION OF PROBLEMS

26. The sum of two numbers is 28, and their difference is 4. Find the numbers.

Let $x =$ the greater, 16

then $28 - x =$ the less, 12

$-28 + 2x =$ the difference

$4 =$ the difference

$\therefore\ -28 + 2x = 4$

$2x = 4 + 28 = 32$

$x = 16$

PROOF

1. $16 + 12 = 28$
2. $16 - 12 = 4$

27. Solve Ex. 26 by letting x equal the smaller.

28. Discover the relation of the terms and solve Ex. 26 indirectly.

Since the sum of two numbers plus their difference is twice the greater, twice the greater is 32, or the greater is 16.

NOTE. After the law is established the solution by the indirect method is the simpler, but considerable ingenuity is required to discover the law.

29. A and B together have \$35; A's money plus \$9 equals B's. How much has each?

30. A man weighs 29 lb. more than his wife; the sum of their weights is 315 lb. What is the weight of each?

31. A and B formed a partnership with a joint capital of \$5000; A's investment exceeded B's by \$200. How much was invested by each?

32. A is 25 years older than B, and A's age is as much above 20 as B's is below 85. Find their ages.

33. A has \$125 and B has \$45. How many dollars must A give B in order that they may have equal amounts?

34. Three boys, A, B, and C, pull 100 pounds; A pulls 20 lb. more than B, and B 8 lb. less than C. How many pounds does each boy pull?

35. The sum of two numbers is a; their difference is b. Find the numbers.

It is often wise to represent the unknown quantity by $2x$, $3x$, $4x$, $\cdots$, instead of by x.

36. Three times A's age is equal to 7 times B's age; 9 years hence 3 times A's will equal 5 times B's. How old is each?

SUGGESTION. Let $7x =$ A's age in years; then $3x$ must equal B's age, because 3 times A's age, or $21x$, equals 7 times B's age.

37. Three times A's age is equal to 4 times B's age; the sum of their ages is 70 years. How old is each?

38. Five times B's money was equal to 11 times A's money. After B gave A $60, 11 times what he had left was equal to 5 times what A then had. How much had each at first?

39. A watch and chain are worth $185, and the watch lacks $19 of being worth twice as much as the chain. Find the value of each. Let $x =$ value of chain in dollars.

40. The sum of three numbers is 263; the first is 3 times the second, and the third is 23 more than 5 times the sum of the other two. Find the numbers.

41. A is 3 times as old as B; 12 years ago he was 5 times as old. How old is each?

42. If twice a number is diminished by 12 and the result is subtracted from 3 times the number, the remainder multiplied by 4 will equal 96 plus twice the number. What is the number?

43. A doubles his capital every year, but at the end of each year deducts $2500 for living expenses; at the end of four years he has 11 times his original stock. How much had he at first?

44. Two men received the same sum for their labor; if one had received $15 less and the other $15 more, one would have received 4 times as much as the other. How much did each receive?

45. A horse and carriage cost $252; the cost of the horse was $12 more than 3 times the cost of the carriage. Find the cost of each.

PRODUCTS AND FACTORS

THE SQUARE OF AN EXPRESSION

Expressions constructed on certain distinct plans occur so frequently in algebraic manipulations that it is a saving of effort to unite them into products or to separate them into factors by methods other than those that have already been discussed. The most important of these forms are: The square of any expression; the product of the sum and difference of two quantities; the product of such expressions as $x \pm a$ and $x \pm b$; the binomial theorem; and the $x^n \pm y^n$ theorem.*

PROPOSITION XXIII. THEOREM

To square any algebraic expression, to the sum of the squares of the several terms add twice the product of each term by each of the terms that follow it.

Let us take $(a - b + c)^2$. By multiplication in full,

$$\begin{array}{llllll}
a & -\ b & +\ c & & & \\
a & -\ b & +\ c & & & \\
\hline
a^2 & -\ ab & +\ ac & & & \\
 & -\ ab & & +\ b^2 & -\ bc & \\
 & & \ \ \ ac & & -\ bc & +\ c^2 \\
\hline
a^2 & -\ 2\,ab & +\ 2\,ac & +\ b^2 & -\ 2\,bc & +\ c^2
\end{array}$$

Examining the result, we see that a^2, b^2, and c^2 are the squares of a, $-b$, and c; that $-2\,ab$, $2\,ac$, and $-2\,bc$ are twice the products of a and $-b$, a and c, and $-b$ and c.

Hence, the principle, Q.E.D.

* $x \pm a$ is read x plus or minus a; $x^n \pm y^n$ is read x to the nth plus or minus y to the nth.

Find the product:

1. $(a-b)^2$
2. $(x+y)^2$
3. $(x-y)^2$
4. $(x+3y)^2$
5. $(2x-3y)^2$
6. $(5a^2-2b^2)^2$
7. $(a-b+c-d)^2$
8. $(a+b+c)^2$
9. $(a-b-c)^2$
10. $(a-b+c)^2$
11. $(a-b-c-d)^2$
12. $(a+b-c+d)^2$

Ex. 1. $(a-b)^2 = a^2 - 2ab + b^2$. The sum of the squares is a^2+b^2; twice the product of the terms, $-2ab$.

Ex. 7. $(a-b+c-d)^2 = a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$. The sum of the squares is $a^2+b^2+\cdots$; twice the products, $-2ab+2ac-\cdots$.

Find the factors:

13. $a^2-2ab+b^2$
14. $x^2+2xy+y^2$
15. $x^2-2xy+y^2$
16. $a^2+8a+16$
17. $a^2-8a+16$
18. $x^4+2x^2y^2+y^4$
19. $a^2-20a+100$
20. $49a^2b^2-56abcd+16c^2d^2$
21. $4x^4y^2-24x^2y^3z+36y^4z^2$
22. $25x^2y^2+60xyz+36z^2$
23. $a^2-2ab-2ac+b^2+c^2+2bc$
24. $a^2+2ab-2ac+b^2+c^2-2bc$
25. $a^2+2ab+2ac+b^2+c^2+2bc$
26. $a^2+b^2+c^2-2ab+2ac-2bc$

Ex. 26.
$$a^2+b^2+c^2-2ab+2ac-2bc$$
$$a,\ -b,\ +c$$
$$(a-b+c)(a-b+c)$$

The square root of a^2 is a; of b^2, $-b$; of c^2, c. Twice the product of a and $-b$ is $-2ab$; of a and c, $2ac$; etc.

NOTE. The square root of a^2 is $+a$ or $-a$; of b^2, $+b$ or $-b$. It is customary to consider the first root positive and to give other roots the signs needed for the products.

PRODUCT OF SUM AND DIFFERENCE

Proposition XXIV. Theorem

The product of the sum and difference of two quantities is the difference of their squares.

Let us take the quantities a and b; their sum is $a+b$, their difference, $a-b$. By multiplication in full,

$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\quad -ab-b^2} \\ a^2 \qquad\quad -b^2 \end{array}$$

Examining the result, we see that a^2 is the square of a; that $-b^2$ is minus the square of b.

Hence, the principle, Q.E.D.

Find the product:

27. $(2a+3b)(2a-3b)$
28. $(x+y)(x-y)$
29. $(x^2+y^2)(x^2-y^2)$
30. $(6x^2+y^2)(6x^2-y^2)$
31. $(3x+2y)(3x-2y)$
32. $(2x+y)(2x-y)$
33. $(3x+2y)(3x-2y)$
34. $(a+b-c)(a-b-c)$
35. $(a+b+c)(a-b+c)$
36. $(a-b+c)(a-b-c)$
37. $(a^2-ab+b^2)(a^2+ab+b^2)$
38. $(a^2+a+1)(a^2-a+1)$
39. $(x^2+x+1)(x^2-x+1)$
40. $(x^2+xy+y^2)(x^2-xy+y^2)$

Ex. 27. $(2a+3b)(2a-3b)$
$2a,\ 3b$
$4a^2-9b^2$

The quantities are $2a$ and $3b$; the product is $4a^2-9b^2$.

Ex. 34. $(a+b-c)(a-b-c)$
$a-c,\ b$
$a^2-2ac+c^2-b^2$

The quantities are $a-c$ and b; the product is $a^2-2ac+c^2-b^2$.

Note. The first quantity must be made up of the terms which have the same sign in both factors.

Find the factors:

41. $a^2 - b^2$
42. $x^2 - y^2$
43. $a^2 - 4$
44. $b^2 - 16$
45. $a^4 - b^4$
46. $a^6 - b^6$
47. $a^8 - b^8$
48. $a^8 - 1$
49. $4x^2 - 9y^2$
50. $1 - 36a^2$
51. $9a^2 - 25b^2$
52. $36x^2 - 49y^2$
53. $x^2 - (a - b)^2$
54. $(a + b)^2 - c^2$
55. $4a^2c^2 - (a + b + c)^2$
56. $(a + b)^2 - (c + d)^2$
57. $(a - b)^2 - (c - d)^2$
58. $x^2 + y^2 - z^2 - 2xy$
59. $x^2 - y^2 + z^2 + 2xz$
60. $a^2 - b^2 + m^2 - n^2 + 2am + 2bn$
61. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$
62. $a^2 + b^2 - c^2 - d^2 - 2ab + 2cd$
63. $a^2 - 2ay + y^2 - x^2 - 2xz - z^2$
64. $a^2 + 2ay + y^2 - x^2 - 2xz - z^2$

Ex. 41.

$$a^2 - b^2$$
$$a, \quad b$$
$$(a + b)(a - b)$$

This is the product of the sum and difference of two quantities. The quantities are a and b; their sum is $a + b$; their difference is $a - b$.

Ex. 60.

$$a^2 - b^2 + m^2 - n^2 + 2am + 2bn$$
$$(a^2 + 2am + m^2) - (b^2 - 2bn + n^2) \qquad (1)$$
$$a + m, \qquad b - n$$
$$(a + m + b - n) \quad (a + m - b + n)$$

Arranging this in the form of the difference of two squares to get the product of the sum and difference of two quantities, we have (1). The quantities are $a + m$ and $b - n$; their sum is $a + m + b - n$; their difference is $a + m - b + n$.

NOTE. The perfect squares may be selected by observing that $2am$ must go with a^2 and m^2, and that $2bn$ must go with b^2 and n^2.

PRODUCT OF SUM AND DIFFERENCE

Find the factors:

65. $4x^2y^2-(x^2+y^2-z^2)^2$
66. $(a^2-b^2-c^2)^2-4b^2c^2$
67. $4a^2b^2-(a^2+b^2+c^2)^2$
68. $4a^2b^2-(a^2-b^2-c^2)^2$
69. $4a^2c^2-(a^2-b^2+c^2)^2$
70. $x^4+x^2y^2+y^4$
71. $a^4+a^2b^2+b^4$
72. $1+4x^4$
73. $4a^4-28a^2b^2+9b^4$
74. $4a^4-48a^2b^2+9b^4$

Ex. 65.

$$4x^2y^2-(x^2+y^2-z^2)^2$$
$$2xy, \qquad x^2+y^2-z^2$$
$$(2xy+x^2+y^2-z^2)(2xy-x^2-y^2+z^2)$$
$$[(x^2+2xy+y^2)-z^2][z^2-(x^2-2xy+y^2)] \qquad (1)$$
$$x+y,\ z \qquad z,\ x-y$$
$$(x+y+z)(x+y-z)(z+x-y)(z-x+y)$$

This is the product of the sum and difference of two quantities. The quantities are $2xy$ and $x^2+y^2-z^2$; their sum is $2xy+x^2+y^2-z^2$; their difference is $2xy-x^2-y^2+z^2$.

Arranging these factors in the form of the difference of two squares, we have (1). The quantities of the first factor are $x+y$ and z; of the second factor, z and $x-y$; their sums and differences respectively are $x+y+z$, $x+y-z$, and $z+x-y$, $z-x+y$.

NOTE. The square brackets are necessary because the two factors are to be multiplied.

Ex. 70.

$$x^4+x^2y^2+y^4$$
$$(x^4+2x^2y^2+y^4)-x^2y^2 \qquad (2)$$
$$x^2+y^2, \qquad xy$$
$$(x^2+xy+y^2)(x^2-xy+y^2)$$

Arranging this in the form of the difference of two squares, we have (2). The quantities are x^2+y^2 and xy; their sum is x^2+xy+y^2; their difference is x^2-xy+y^2.

NOTE. The form $x^4+x^2y^2+y^4$ should be carefully analyzed. Exs. 71 to 74 may be solved in the same way.

THE $(x \pm a)(x \pm b)$ THEOREM

PROPOSITION XXV. THEOREM

To multiply such expressions as $x \pm a$ and $x \pm b$, for the first term write the product of the two first terms; for the second term, write the sum of the two last terms multiplied by the first term; for the third term, write the product of the two last terms.

Let us take $x + 5$ and $x - 3$. By multiplication in full,

$$\begin{array}{l} x + 5 \\ \underline{x - 3} \\ x^2 + 5x \\ \underline{\quad\;\; - 3x - 15} \\ x^2 + 2x - 15 \end{array}$$

Examining the result, we see that x^2 is the product of x and x; that $2x$ is the sum of $+5$ and -3 multiplied by x; that -15 is the product of $+5$ and -3.

Hence, the principle, Q.E.D.

Find the product:

75. $(x - 5)(x + 8)$

76. $(x - 3)(x - 7)$

77. $(x - 9)(x + 8)$

78. $(x + 8)(x + 5)$

79. $(x^2 - 3)(x^2 + 5)$

80. $(x - 2y)(x + 3y)$

81. $(ax - 5y)(ax + 8y)$

82. $(bx - y)(bx + 3y)$

83. $(2x - 3b)(2x + 2b)$

84. $(3a^2 - 3b^2)(3a^2 + 4b^2)$

85. $(3a + 5b)(3a - 3b)$

86. $(4x^2 + 2y^2)(4x^2 - 7y^2)$

Ex. 81.

$$(ax - 5y)(ax + 8y)$$
$$a^2x^2 + 3axy - 40y^2$$

The product of the two first terms is a^2x^2; the sum of the two last terms multiplied by the first is $3axy$; the product of the two last terms is $-40y^2$.

THE $(x \pm a)(x \pm b)$ THEOREM

Find the factors:

87. $x^2 - 10x + 16$

88. $x^2 + 7x + 12$

89. $a^2 - 4a + 3$

90. $a^2 - 7a + 10$

91. $x^2 + 6x - 7$

92. $a^2 - 3a - 28$

93. $y^2 - 7y - 18$

94. $a^2 + 5a - 84$

95. $a^2 - 7a + 12$

96. $b^2 - 9b - 36$

97. $a^2x^2 - 10abx + 16b^2$

98. $a^2 - 3ab - 10b^2$

99. $x^2 + 3xy + 2y^2$

100. $a^2b^2 + 11abc + 30c^2$

101. $a^2 - abc - 30b^2c^2$

102. $1 - abc - 30a^2b^2c^2$

103. $a^2b^2c^2 - abc - 30$

104. $a^2b^2 + 7abx + 10x^2$

105. $a^6 + 3a^3 - 4$

106. $c^8 + 15c^4 - 100$

107. $(a - b)^2 - 10(a - b) + 16$

108. $(x - y)^2 + 3(x - y) - 10$

Ex. 87.

$$x^2 - 10x + 16$$
$$-2, \quad -8$$
$$(x - 2)(x - 8)$$

We are to find two numbers whose product is 16 and whose sum is -10; the numbers are -2 and -8; the factors are $x - 2$ and $x - 8$.

Ex. 107.

$$(a - b)^2 - 10(a - b) + 16$$
$$-2, \quad -8$$
$$(a - b - 2)(a - b - 8)$$

We are to find two numbers whose product is 16 and whose sum is -10; the numbers are -2 and -8; the factors are $a - b - 2$ and $a - b - 8$.

A monomial is an algebraic expression of one term; a binomial, of two terms; a trinomial, of three terms.

Thus, a is a monomial; $a + b$ is a binomial; $a + b + c$ is a trinomial.

The binomial theorem is the statement of laws by which a binomial may be raised to any power.

THE BINOMIAL THEOREM

Proposition XXVI. Theorem

The exponent of the leading letter in the first term of the product is the same as the exponent of the binomial and decreases by one in each succeeding term.

*The exponent of the second letter in the first term of the product is zero and increases by one in each succeeding term.**

The coefficient of the first term is one; the coefficient of the second term is the same as the exponent of the binomial; the coefficient of each succeeding term may be found by multiplying the coefficient of the preceding term by the exponent of the leading letter, and dividing by the exponent of the second letter increased by one.

If the second term of the binomial is minus, the sign of every alternate term in the product, beginning with the second, is changed.

$$
\begin{array}{lcl}
a+b & (a\pm b)^1 & a-b \\
a^2+2ab+b^2 & (a\pm b)^2 & a^2-2ab+b^2 \\
\underline{\qquad a+b\qquad} & & \underline{\qquad a-b\qquad} \\
a^3+2a^2b+ab^2 & & a^3-2a^2b+ab^2 \\
\underline{\qquad a^2b+2ab^2+b^3} & & \underline{\quad -a^2b+2ab^2-b^3} \\
a^3+3a^2b+3ab^2+b^3 & (a\pm b)^3 & a^3-3a^2b+3ab^2-b^3 \\
\underline{\qquad\qquad a+b\quad} & & \underline{\qquad\qquad a-b\quad} \\
a^4+3a^3b+3a^2b^2+ab^3 & & a^4-3a^3b+3a^2b^2-ab^3 \\
\underline{\qquad a^3b+3a^2b^2+3ab^3+b^4} & & \underline{\quad -a^3b+3a^2b^2-3ab^3+b^4} \\
a^4+4a^3b+6a^2b^2+4ab^3+b^4 & (a\pm b)^4 & a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots
\end{array}
$$

Hence, the principles, Q.E.D.

* The second letter does not appear in the first term of the product; hence, its exponent is 0. $a+b$ may be written ab^0+a^0b; $a^2+2ab+b^2$ may be written $a^2b^0+2ab+a^0b^2$; and so on. See Prop. VIII., p. 14.

THE BINOMIAL THEOREM

Find the product:

109. $(a+b)^3$

110. $(a-b)^5$

111. $(a-b)^7$

112. $(a+b)^6$

113. $(a-b)^8$

114. $(x+y)^{10}$

115. $(1+a)^4$

116. $(a-1)^6$

117. $(2x+3y)^4$

118. $(m+2n)^5$

119. $(2x-3y)^3$

120. $(a+b)^m$

Ex. 110. $(a-b)^5$

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$

We proceed as if the base were $a+b$, and then change the sign of each alternate term beginning with the second.

In the development of $(a+b)^3$ on p. 55, we observe that when the exponent is odd the last half of the coefficients are the same as the first half in reverse order. Therefore, we may compute the first set of coefficients and write them in reverse order for the last set.

Ex. 112. $(a+b)^6$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

In the development of $(a+b)^4$ on p. 55, we observe that when the exponent is even, the coefficients of the terms after the middle term are the same as those before this term in reverse order. Therefore, we may compute the first set of coefficients and write them in reverse order for the last set.

Ex. 115. $(1+a)^4$

$$(1+a)^4 = 1 + 4a + 6a^2 + 4a^3 + a^4$$

Since 1 to any power is 1, it is not necessary to write the exponents of 1.

Ex. 117. $(2x+3y)^4$

$(2x+3y)^4 = (2x)^4 + 4(2x)^3(3y) + 6(2x)^2(3y)^2 + 4(2x)(3y)^3 + (3y)^4$, or $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$.

Whatever is within a parenthesis is regarded as a base. $2x$ is for a; $3y$ is for b.

Find the factors:

121. $a^3 - 3a^2b + 3ab^2 - b^3$ **122.** $a^3 + 3a^2b + 3ab^2 + b^3$

123. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

124. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$

125. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

126. $a^8 + 8a^7 + 28a^6 + 56a^5 + 70a^4 + 56a^3 + 28a^2 + 8a + 1$

Ex. 121.

$$a^3 - 3a^2b + 3ab^2 - b^3$$
$$a, \quad -b$$
$$(a-b)^3$$

The cube root of a^3 is a; of $-b^3$ is $-b$; the factors are $a-b$, $a-b$ and $a-b$; the cube of $a-b$ is the expression.

THE $x^n \pm y^n$ THEOREM

Since the exponents of $x^n \pm y^n$ must be either odd or even, four types are possible. These are given below with the lowest type of each first.

I. $x+y$, x^3+y^3, x^5+y^5, $\cdots$ exponents odd, sign '+'
II. $x-y$, x^3-y^3, x^5-y^5, $\cdots$ exponents odd, sign '−'
III. x^2+y^2, x^4+y^4, x^6+y^6, $\cdots$ exponents even, sign '+'
IV. x^2-y^2, x^4-y^4, x^6-y^6, $\cdots$ exponents even, sign '−'

The laws for the divisibility of these types are often stated as below, but it is better to merge them into one law, as in Prop. XXVII.

The sum of the same odd powers of two numbers is divisible by their sum. The difference of the same odd powers of two numbers is divisible by their difference. The sum of the same even powers of two numbers is divisible by neither their sum nor their difference. The difference of the same even powers of two numbers is divisible by either their sum or their difference.

THE $x^n \pm y^n$ THEOREM

PROPOSITION XXVII. THEOREM

$x^n \pm y^n$ is divisible by $x \pm y$ if the lowest type is so divisible.

The exponent of the leading letter in the first term of the quotient is one less than the exponent in the dividend, and decreases by one in each succeeding term.

*The exponent of the second letter in the first term of the quotient is zero, and increases by one in each succeeding term.**

The coefficient of each term is one.

If the divisor is $x+y$, the signs of the quotient are alternately '+' and '−'; if the divisor is $x-y$, the signs of the quotient are all '+.'

Let us divide $x^n + y^n$ by $x + y$ and then by $x - y$

$$
\begin{array}{l}
x+y)\,x^n + y^n\,(x^{n-1} - x^{n-2}y \\
\qquad \underline{x^n + x^{n-1}y} \\
\qquad\quad - x^{n-1}y + y^n \\
\qquad\quad \underline{- x^{n-1}y - x^{n-2}y^2} \\
\qquad\qquad\qquad x^{n-2}y^2 + y^n \text{ or} \\
\qquad\qquad\qquad (x^{n-2} + y^{n-2})\,y^2
\end{array}
\qquad
\begin{array}{l}
x-y)\,x^n + y^n\,(x^{n-1} + x^{n-2}y \\
\qquad \underline{x^n - x^{n-1}y} \\
\qquad\quad x^{n-1}y + y^n \\
\qquad\quad \underline{x^{n-1}y - x^{n-2}y^2} \\
\qquad\qquad\qquad x^{n-2}y^2 + y^n \text{ or} \\
\qquad\qquad\qquad (x^{n-2} + y^{n-2})\,y^2
\end{array}
$$

Thus, $x^n + y^n$ is divisible by $x \pm y$ if $x^{n-2} + y^{n-2}$ is so divisible; $x^{n-2} + y^{n-2}$ is divisible by $x \pm y$ if $x^{n-4} + y^{n-4}$ is so divisible; and so on to the lowest type. That is, $x^n + y^n$ is divisible by $x \pm y$ if the lowest type is so divisible.

* The second letter does not appear in the first term of the quotient; hence, its exponent is 0. The quotient of $x^6 - y^6$ by $x - y$ may be written $x^5y^0 + x^4y + x^3y^2 + x^2y^3 + xy^4 + x^0y^5$. See Prop. VIII, p. 14.

Let us divide $x^n - y^n$ by $x + y$ and then by $x - y$

$$\begin{array}{l}
x + y)\,x^n - y^n\,(x^{n-1} - x^{n-2}y \\
\qquad \underline{x^n + x^{n-1}y} \\
\qquad\quad -x^{n-1}y - y^n \\
\qquad\quad \underline{-x^{n-1}y - x^{n-2}y^2} \\
\qquad\qquad\qquad x^{n-2}y^2 - y^n \text{ or} \\
\qquad\qquad\qquad (x^{n-2} - y^{n-2})\,y^2
\end{array}
\qquad
\begin{array}{l}
x - y)\,x^n - y^n\,(x^{n-1} + x^{n-2}y \\
\qquad \underline{x^n - x^{n-1}y} \\
\qquad\quad x^{n-1}y - y^n \\
\qquad\quad \underline{x^{n-1}y - x^{n-2}y^2} \\
\qquad\qquad\qquad x^{n-2}y^2 - y^n \text{ or} \\
\qquad\qquad\qquad (x^{n-2} - y^{n-2})\,y^2
\end{array}$$

Thus, $x^n - y^n$ is divisible by $x \pm y$ if $x^{n-2} - y^{n-2}$ is so divisible; $x^{n-2} - y^{n-2}$ is divisible by $x \pm y$ if $x^{n-4} - y^{n-4}$ is so divisible; and so on to the lowest type. That is, $x^n - y^n$ is divisible by $x \pm y$ if the lowest type is so divisible.

If the division in each of the four cases is continued, the quotients are:

$$(x^n \pm y^n) \div (x + y) = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4 - \cdots$$
$$(x^n \pm y^n) \div (x - y) = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + x^{n-5}y^4 + \cdots$$

Hence, the principles, Q.E.D.

ILLUSTRATIONS:

$$\begin{aligned}
x^5 + y^5 &= (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \\
x^5 - y^5 &= (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \\
x^6 - y^6 &= (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) \\
&= (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)
\end{aligned}$$

$x^5 + y^5$ is divisible by $x + y$ because the lowest type, $x + y$, is so divisible; $x^5 - y^5$ is divisible by $x - y$ because the lowest type, $x - y$, is so divisible; $x^6 - y^6$ is divisible by either $x + y$ or $x - y$ because the lowest type, $x^2 - y^2$, is so divisible; $x^6 + y^6$ is not divisible by $x + y$ nor by $x - y$ because the lowest type, $x^2 + y^2$, is not so divisible.

THE $x^n \pm y^n$ THEOREM

Find the factors:

127. $x^3 + y^3$

128. $x^3 - y^3$

129. $x^4 - y^4$

130. $x^5 + y^5$

131. $x^5 - y^5$

132. $x^6 - y^6$

133. $x^7 + y^7$

134. $x^7 - y^7$

Ex. 129. $x^4 - y^4 = (x + y)(x^3 - x^2y + xy^2 - y^3)$, or $(x - y)(x^3 + x^2y + xy^2 + y^3)$. By the method of p. 62 the second factor of each set may be further resolved. *Ans.* $(x^2 + y^2)(x + y)(x - y)$.

Find the product:

135. $(x + y)(x^2 - xy + y^2)$

136. $(x - y)(x^2 + xy + y^2)$

137. $(x + y)(x^3 - x^2y + xy^2 - y^3)$

138. $(x - y)(x^3 + x^2y + xy^2 + y^3)$

139. $(x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5)$

140. $(x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)$

Ex. 135. $(x + y)(x^2 - xy + y^2) = x^3 - y^3$. Observe that the value of $(x + y)(x^2 + xy + y^2)$ and the value of $(x - y)(x^2 - xy + y^2)$ cannot be found in this way.

The difference of the same even powers of two numbers may be factored more readily by regarding the binomial as the product of the sum and difference of two quantities.

Find the factors:

141. $x^4 - y^4$

142. $x^6 - y^6$

143. $x^8 - y^8$

144. $x^{10} - y^{10}$

Ex. 141. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$. See Ex. 129.

Ex. 144. $x^{10} - y^{10} = (x^5 + y^5)(x^5 - y^5) = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.

Some binomials may be changed to the form $x^n \pm y^n$ and then factored.

Find the factors:

145. $16a^4 - 81b^4$ **149.** $1 + x^5$ **153.** $x^2 - 1$

146. $32a^5 + 243b^5$ **150.** $1 - m^3$ **154.** $x^7 + 1$

147. $16x^4 - 81$ **151.** $1 + m^3$ **155.** $x^5 - 1$

148. $27x^3 - 64y^3$ **152.** $1 - x^4$ **156.** $x^6 - 1$

Ex. 145. $16a^4 - 81b^4 = (2a)^4 - (3b)^4$

$$= [2a - 3b] ; [(2a)^3 + (2a)^2(3b) + (2a)(3b)^2 + (3b)^3]$$
$$= (2a - 3b)(8a^3 + 12a^2b + 18ab^2 + 27b^3).$$

A second solution. The first term of the second factor may be found by dividing the first term of the binomial by the first term of the first factor. Each succeeding term may be found by dividing the preceding term by the first term of the factor and multiplying the result by the second term of the factor. Thus, $16a^4 \div 2a = 8a^3$; $(8a^3 \div 2a) \times 3b = 12a^2b$; $(12a^2b \div 2a) \times 3b = 18ab^2$; $(18ab^2 \div 2a) \times 3b = 27b^3$.

Ex. 149. $1 + x^5 = 1^5 + x^5 = (1 + x)(1 - x + x^2 - x^3 + x^4)$

The sum of the same even powers of two numbers can be factored when the exponent has an odd factor other than 1.

Find the factors:

157. $x^6 + y^6$ **160.** $x^6 + 1$ **163.** $x^{14} + y^{14}$

158. $x^{10} + y^{10}$ **161.** $x^{10} + 1$ **164.** $x^{18} + y^{18}$

159. $x^{12} + y^{12}$ **162.** $x^{12} + 1$ **165.** $x^{20} + y^{20}$

Ex. 157. $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$; $6 = 2 \times 3$; $x^6 + y^6 = (x^2)^3 + (y^2)^3$.

Ex. 159. $x^{12} + y^{12} = (x^4 + y^4)(x^8 - x^4y^4 + y^8)$; $12 = 4 \times 3$; $x^{12} + y^{12} = (x^4)^3 + (y^4)^3$.

NOTE. Observe that the binomial factor has the even exponent and that the exponents of the second factor increase and decrease by this even number.

OTHER CASES — FACTORS

There may be a common factor in all of the terms. It may be found by inspection.

Factor:

166. $7a^3 - 7b^3$

167. $4a^2 - 12ab + 16b^2$

168. $3a^2 - 12ab + 12b^2$

169. $6a^2 - 12ab + 6b^2$

170. $12x^3y - 24x^2y^2 + 36xy^3$

171. $12x^4 + 12x^2y^2 + 12y^4$

172. $6a^5 + 6b^5$

173. $3x^2 + 6x - 105$

174. $8ax^2 - 24axy + 18ay^2$

175. $10x^3y^3 - 20x^2y^4 + 30x^4y^2$

Ex. 170.

$$12x^3y - 24x^2y^2 + 36xy^3$$
$$12xy(x^2 - 2xy + 3y^2)$$

$12xy$ is contained in $12x^3y$, x^2 times; in $-24x^2y^2$, $-2xy$ times; in $36xy^3$, $3y^2$ times.

There may be a common factor in at least two of the terms.

Factor:

176. $a^3 - a^2b - ab^2 + b^3$

177. $ax + ay - xy - y^2$

178. $by + bx - xy - x^2$

179. $abc^2 - axc + bxc - x^2$

180. $ax + by - bx - ay$

181. $a^2x^2 + a^2y^2 - b^2x^2 - b^2y^2$

182. $x^2 + bx + ax + ab$

183. $x^4 + x^3y + xz^3 + yz^3$

184. $x^2y^2 - x^2z^2 - y^2z^2 + z^4$

185. $a^4 - a^3b - ab^3 + b^4$

186. $6a^2x^2 - 4a^2y^2 - 3bx^2 + 2by^2$

187. $xy - 2mn + 2my - nx$

Ex. 176.

$$a^3 - a^2b - ab^2 + b^3$$
$$a^2(a - b) - b^2(a - b)$$
$$(a - b)(a^2 - b^2)$$
$$(a - b)(a + b)(a - b)$$

a^2 is contained in a^3, a times; in a^2b, $-b$ times; $-b^2$ is contained in $-ab^2$, a times; in $+b^3$, $-b$ times.

$a - b$ is contained in $a^2(a - b)$, a^2 times; in $-b^2(a - b)$, $-b^2$ times. $a^2 - b^2$ may be resolved into the factors $a + b$ and $a - b$.

MISCELLANEOUS

Perform the indicated operation:

188. $(1 + 3\,ax)(1 - 3\,ax)$

189. $(3\,x^n - y^n)(3\,x^n + y^n)$

190. $\dfrac{x^7 + b^7}{x + b}$

191. $\dfrac{x^5 + 32}{x + 2}$

192. $(x - y)^{10}$

193. $(4\,a - 2\,b)(4\,a + 2\,b)$

194. $(4\,m^3 + 6\,n)(4\,m^3 - 6\,n)$

195. $\dfrac{x^5 + y^5}{x + y}$

196. $\dfrac{x^3 - 27}{x - 3}$

197. $(a + b)^8$

198. $(x^2 + 2\,x - 1)(x^2 - 2\,x - 1)$

199. $(a^{m+n} + b^{m+n})(a^{m+n} - b^{m+n})$

200. $(a + b)(a^2 - ab + b^2)$

201. $(a - b)(a^3 + a^2b + ab^2 + b^3)$

202. $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$

203. $(a^2 + ab + b^2)(a^2 - ab + b^2)(a + b)(a - b)$

204. $(a - c)(a + b)$

205. $(7 - a)(3 - a)$

206. $(a + 8\,x^3)(a - 8\,x^3)$

207. $(x + y - z)(x + y + z)$

208. $(a + b + 3)(a + b - 7)$

209. $(x - y + 3\,z)(x - y - 5\,z)$

210. $(5\,x^2 + 4)(5\,x^2 - 4)$

211. $(a + b)(a + b)^3$

212. $(a - b)^2(a - b)$

213. $(a^2 + b^2)(a^2 - b^2)$

214. $(a - b + c)^2$

215. $(a - b - c - d)^2$

MISCELLANEOUS

Factor:

216. $x^2 + (a - c)x - ac$

217. $x^5 + x^4 - 5x^3 - 5x^2$

218. $a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2$

219. $(b^2 + 24b + 144) - 121$

220. $a^2 + 22ab + 121b^2$

221. $25a^2 - 4(b + c)^2$

222. $y^4 - (a^2 + b^2)y^2 + a^2b^2$

223. $(a^2+3a)^2-14(a^2+3a)+40$

224. $(a^2 - b^2)^2 - m^2 - 2mn - n^2$

225. $a^2 + 2ay + y^2 + 2a + 2y$

226. $a^2b^2 + 4abc - 45c^2$

227. $(a - b)^2 - (c - d)^2$

228. $2mn - m^2 - n^2 + a^2 + b^2 - 2ab$

229. $64a^3 + 240a^2b + 300ab^2 + 125b^3$

230. $8a^2n^5x^5 - 12an^4x^7 + 4a^2n^3x^8$

231. $(m^2 + 3m)^2 - 14(m^2 + 3m) + 40$

232. $a^2 + b^2 - c^2 - d^2 + 2ab + 2cd$

233. $x^2 - 2xy + y^2 + 9x - 9y + 18$

234. $ax^2 + axy - bx^2 - bxy$

235. $x^3 + 3x^2 + 3x + 1$

236. $a(a - c) - b(b - c)$

237. $9x^4 - 40x^2y^2 + 16y^4$

238. $9x^4 - 88x^2y^2 + 16y^4$

239. $9x^4 + 24x^2y^2 + 16y^4$

240. $28x^4y + 64x^3y - 60x^2y$

241. $x^4 - 2x^2y^2 + y^4$

242. $8(x + y)^3 - (2x - y)^3$

243. $x^6 + 2x^3y^3 + y^6$

244. $(x^2 + x - 4)^2 - 4$

245. $3x^3 - 12x^3y^2 - 4y^2 + 1$

DIVISIBILITY OF NUMBERS

Rules for the divisibility of numbers (integers) may be developed by the use of letters for numbers. The method of procedure is to find a literal expression for any number in the decimal system, to separate it into two parts such that the second part shall be divisible by the number in question, and to interpret the first part. Their proof is based upon the first of two self-evident truths:

1. A factor of each of two numbers is a factor of their sum.
2. A factor of each of two numbers is a factor of their difference.

PROPOSITION XXVIII. THEOREM

Any number in the decimal system may be represented by $a + 10b + 100c + 1000d + \cdots$ *where a is the digit in units' order; b, the digit in tens' order; and so on.*

Every number in the decimal system is equal to its digit in units' order, plus 10 times its digit in tens' order, plus 100 times its digit in hundreds' order, and so on.

Hence, the principle, Q.E.D.

PROPOSITION XXIX. THEOREM

A number is divisible by 2 *when the number denoted by its last digit is divisible by* 2.

Let $a + 10b + 100c + 1000d + \cdots =$ any number.

Let us separate the number into two parts such that the second part shall be divisible by 2.

$$a \mid + 10b + 100c + 1000d + \cdots = \text{any number}$$

Since the second part is divisible by 2, the whole is divisible by 2 if the first part is also divisible by 2.

(A factor of each of two numbers is a factor of their sum.)

The first part is the number denoted by the last digit.

Hence, the principle, Q.E.D.

DIVISIBILITY OF NUMBERS

PROPOSITION XXX. THEOREM

A number is divisible by 3 *when the sum of its digits is divisible by* 3.

Let $a + 10\,b + 100\,c + 1000\,d + \cdots =$ any number.

Let us separate the number into two parts such that the second part shall be divisible by 3. $10\,b = b + 9\,b$; $100\,c = c + 99\,c$; every subsequent term can be separated into two parts, of which the second part is divisible by 3 and the first part is the base taken once, because 1 with any number of ciphers is 1 more than a multiple of 3. That is,

$$a + b + c + d + \cdots \mid + 9\,b + 99\,c + 999\,d + \cdots = \text{any number}$$

Since the second part is divisible by 3, the whole is divisible by 3 if the first part is also divisible by 3.

(A factor of each of two numbers is a factor of their sum.)

The first part is the sum of the digits.

Hence, the principle, Q.E.D.

PROPOSITION XXXI. THEOREM

A number is divisible by 4 *when the number denoted by its last two digits is divisible by* 4.

Let $a + 10\,b + 100\,c + 1000\,d + \cdots =$ any number.

Let us separate the number into two parts such that the second part shall be divisible by 4. $100\,c$ and each subsequent term is divisible by 4. That is,

$$a + 10\,b \mid + 100\,c + 1000\,d + \cdots = \text{any number}$$

Since the second part is divisible by 4, the whole is divisible by 4 if the first part is also divisible by 4. The first part is the number denoted by the last two digits.

Hence, the principle, Q.E.D.

Proposition XXXII. Theorem

A number is divisible by 11 *when the difference between the sum of its digits in the even orders and the sum of its digits in the odd orders is divisible by* 11.

Let $a + 10b + 100c + 1000d + \cdots =$ any number.

Let us separate the number into two parts such that the second part shall be divisible by 11. $10b = 11b - b$; $100c = 99c + c$; every subsequent term can be separated into two parts, the second of which is divisible by 11 and the other alternately '-1' and '$+1$' times the base, because 1 with an odd number of ciphers lacks 1 of being a multiple of 11 and 1 with an even number of ciphers exceeds by 1 a multiple of 11. That is,

$$a - b + c - d + \cdots \mid + 11b + 99c + 1001d + \cdots = \text{any no., or}$$

$$(a+c+\cdots)-(b+d+\cdots) \mid +11b+99c+1001d+\cdots = \text{any no.}$$

Since the second part is divisible by 11, the whole is divisible by 11 if the first part is also divisible by 11. The first part is the difference between the sum of the digits in the even orders and the sum of the digits in the odd orders.

Hence, the principle, Q.E.D.

1. Prove that a number is divisible by 5 when the number denoted by its last digit is divisible by 5.
2. Prove that a number is divisible by 25 when the number denoted by its last two digits is divisible by 25.
3. Prove that a number is divisible by 8 when the number denoted by its last three digits is divisible by 8.
4. Prove that a number is divisible by 9 when the sum of its digits is divisible by 9.

Note. 0 is regarded as being divisible by every number. 0 has the nature of an even number as appears by naming the even numbers backwards: 8, 6, 4, 2, 0, -2, -4 ….

COMMON FACTORS AND MULTIPLES

HIGHEST COMMON FACTOR

"Highest common factor" (H. C. F.) is used with letters instead of "greatest common factor" as in arithmetic, because it is impossible to tell which of two algebraic expressions is the greater.

Thus, $a^3 + b^3$ is higher than $a^2 + 2ab + b^2$ because a^3 has a higher exponent than a^2, but it is impossible to declare which is the greater.

PROPOSITION XXXIII. THEOREM

The highest common factor of two or more quantities is the product of all their common prime factors, each taken the greatest number of times it is found in all of them.

By definition, the H. C. F. of two or more quantities is the highest expression that is contained in each of them without a remainder. Thus, the H. C. F. of a^2b^3 and a^3b^2, or of *aabbb* and *aaabb*, is *aabb*, or a^2b^2.

Hence, the principle, Q.E.D.

It follows that it is necessary to resolve expressions into their prime factors in order to find their H. C. F. The subject naturally presents itself in two cases, where the expressions can be factored by inspection and where they cannot be factored by inspection.

FACTORS BY INSPECTION

Find the H. C. F.:

1. $18a^2b^3,\ 24a^3b^4,\ 36a^2b^5$
2. $24a^2bx^4,\ 42ax^3,\ 18a^3x^2y$
3. $a^2+2ab+b^2,\ a^4-b^4,\ a^4-2a^2b^2+b^4$
4. $x^2-3x+2,\ x^2-2x+1,\ x^2-1$
5. $a^2b^2x^3,\ a^3b^3x^4,\ a^4b^2x^5,\ a^2bx^2$
6. $a^3-b^3,\ a^2-2ab+b^2,\ a^2-b^2$
7. $x^2-2x-24,\ x^2+9x+20$
8. $12ax^2(a-b),\ 16a^2x^3(a^2-b^2),\ 8a^3x^3(a^2-2ab+b^2)$
9. $a^2-b^2,\ a^2-ab,\ a^2-2ab+b^2$
10. $x^2+5x+6,\ x^2+7x+10,\ x^2+12x+20$
11. $x^2-1,\ (x-1)^2,\ x^3-1$
12. $x^2-3x,\ x^2-9,\ x^2-6x+9$
13. $x^2-1,\ x^4-1,\ x^3-2x^2+x$
14. $a^2-ab+ac-bc,\ a^2-ab-ac+bc,\ ac-bc$
15. $2x^2+10x+12,\ 3x^2-3x-18,\ 4x^2+24x+32$
16. $a^2x^2-a^2,\ 2ax^2+2ax-4a,\ 4ax^2-12ax+8a$

Ex. 1. The H. C. F. of the numerical parts is 6; of the a's, a^2; of the b's, b^3; of the whole expressions, $6a^2b^3$.

Ex. 3.

$$a^2+2ab+b^2=(a+b)^2$$

$$a^4-b^4=(a^2+b^2)(a+b)(a-b)$$

$$a^4-2a^2b^2+b^4=(a+b)^2(a-b)^2$$

The H.C.F. of the $(a+b)$'s is $a+b$; of the whole expressions, $a+b$.

H. C. F. — NO FACTORS BY INSPECTION

When no common factor can be found by inspection, it is necessary to seek for lower expressions which have the same highest common factor because lower expressions are easier to analyze. They are found by means of the next three propositions.

Proposition XXXIV. Theorem

Multiplying one of two expressions by a number prime to the other does not affect their H. C. F.

Multiplying the first by a number prime to the second cannot introduce into the first any factor found in the second, and cannot, therefore, affect their H. C. F.

Hence, the principle, Q.E.D.

17. Find expressions having the same H. C. F. as $2x^2 - 7x + 3$ and $3x^2 - 7x - 6$.

Ans. $2x^2 - 7x + 3$ and $6x^2 - 14x - 12$. 2 is prime to $2x^2 - 7x + 3$; $\therefore 2x^2 - 7x + 3$ and $(3x^2 - 7x - 6) \times 2$ have the same H. C. F. as the original expressions.

Note. These results are not lower than the original expressions, but this process is a step toward such a condition. See p. 72.

Proposition XXXV. Theorem

Dividing one of two expressions by a number prime to the other does not affect their H. C. F.

Dividing the first by a number prime to the second cannot take from the first any factor found in the second, and cannot, therefore, affect the H. C. F.

Hence, the principle, Q.E.D.

18. Find lower expressions having the same H. C. F. as $2x^2 - 7x + 3$ and $7x - 21$.

Ans. $2x^2 - 7x + 3$ and $x - 3$. 7 is prime to $2x^2-7x+3$; $\therefore 2x^2-7x+3$ and $(7x - 21) \div 7$ have the same H. C. F. as the original expressions. See p. 72.

PROPOSITION XXXVI. THEOREM

The highest common factor of two expressions is the highest common factor of the divisor and of the remainder found by dividing one of the expressions by the other.

Let 6×9 and 17×9 be two expressions of which 9 is the H. C. F. When 17×9 is divided by 6×9 the quotient is 2; then, the remainder must be $(17-12)9$.

To prove that 9 is the H. C. F. of 6×9 and $(17-12)9$.

$$\begin{array}{r} 6 \times 9)\ 17 \times 9(2 \\ \underline{12 \times 9} \\ (17-12)9 \end{array}$$

9 is a common factor of 6×9 and $(17-12)9$ by inspection. It is the H. C. F. if 6 and $17-12$ are prime to each other. This condition is met because 6 is prime to 17 by hypothesis and is a factor of 12.

Let nF and mF be any two expressions of which F is the H. C. F. When mF is divided by nF the quotient may be q; then, the remainder must be $(m-nq)F$.

To prove that F is the H. C. F. of nF and $(m-nq)F$.

$$\begin{array}{r} nF)\,mF(q \\ \underline{nqF} \\ (m-nq)F \end{array}$$

F is a common factor of nF and $(m-nq)F$ by inspection. It is the H. C. F. if n and $m-nq$ are prime to each other. This condition is met because n is prime to m by hypothesis and is a factor of nq.

(If a number is prime to one of two numbers but a factor of the other, it is prime to their sum or to their difference.)

Hence, the principle, Q.E.D.

Hence, the principle, Q.E.D.

NOTE. At the left is given a proof for two particular expressions; at the right, a proof for any two expressions. The particular proof may be made general by the statement. In a similar manner the principle may be demonstrated for any two expressions.

H. C. F. — NO FACTORS BY INSPECTION

Find the H. C. F.:

19. $2x^2-7x+3$ and $3x^2-7x-6$

20. $4x^2-16x+15$ and $4x^2+4x-15$

21. $x^3-5x^2+11x-15$ and $2x^3-7x^2+16x-15$

22. $a^3-4a^2+9a-10$ and $a^3+5a^2-9a+35$

23. a^3-2a^2+2a-1 and a^3-3a^2+3a-2

24. $2a^3-5a^2-5a-7$ and $2a^3-9a^2+5a+7$

25. $2a^3-a^2-5a+3$ and $2a^3-a^2-a-3$

26. $3x^3-x^2-2x-16$ and $2x^3-2x^2-3x-2$

27. $6a^3+13a^2x-9ax^2-10x^3$ and $9a^3+12a^2x-11ax^2-10x^3$

Ex. 19.

$$
\begin{array}{l}
2x^2-7x+3)3x^2-7x-6(\\
2x^2-7x+3)6x^2-14x-12(3 \\
\qquad\qquad\qquad \underline{6x^2-21x+9} \\
\qquad\qquad\qquad\qquad 7x-21 \\
\qquad\qquad\qquad\qquad x-3)2x^2-7x+3(2x-1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^2-6x} \\
\qquad\qquad\qquad\qquad\qquad\qquad -x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-x+3}
\end{array}
$$

The H. C. F. of $2x^2-7x+3$ and $3x^2-7x-6$ is the H. C. F. of $2x^2-7x+3$ and $6x^2-14x-12$ (XXXIV). We do not, for the first step, employ XXXVI because the first term of the dividend, $3x^2$, is not exactly divisible by the first term of the divisor, $2x^2$.

The H. C. F. of $2x^2-7x+3$ and $6x^2-14x-12$ is the H. C. F. of $2x^2-7x+3$ and $7x-21$ (XXXVI).

The H. C. F. of $2x^2-7x+3$ and $7x-21$ is the H. C. F. of $2x^2-7x+3$ and $x-3$ (XXXV).

The H. C. F. of $2x^2-7x+3$ and $x-3$ is $x-3$ (XXXVI). $\therefore$, $x-3$ is the H. C. F. of the original expressions.

L. C. M. — PRINCIPLE

Proposition XXXVII. Theorem

The lowest common multiple (L. C. M.) of two or more expressions is the product of all their prime factors, each taken the greatest number of times it is found in any one of them.

By definition, the L. C. M. of two or more expressions is the lowest expression which will contain each of them without a remainder.

Hence, the principle, Q.E.D.

It is evident that expressions must be resolved into their prime factors as a preparation for finding their L. C. M.

FACTORS BY INSPECTION

Find the L. C. M.:

28. $18a^3b^2c$, $20a^4bc^3$, and $40a^2b^3c^2$

29. $3x^3yz^3$, $5x^2y^3z^2$, $15x^2y^2z$, and $20x^3y^3z^3$

30. $21a^4x^3y$, $35a^2x^4y$, $28a^3x^2y^4$, and $14a^5x^2y^2$

31. $10x^2y^2z$, $20x^3y^2z$, and $25x^2y^3z^3$

32. x^2-y^2, $x^2+2xy+y^2$, x^3+y^3

Ex. 28.

$$18a^3b^2c = 2\cdot 3^2a^3b^2c$$
$$20a^4bc^3 = 2^2\cdot 5a^4bc^3$$
$$40a^2b^3c^2 = 2^3\cdot 5a^2b^3c^2$$

The L. C. M. of the 2's is 2^3; of the 3's, 3^2; of the 5's, 5; of the a's, a^4; of the b's, b^3; of the c's, c^3; of the whole, $2^3\cdot 3^2\cdot 5a^4b^3c^3$, or $360a^4b^3c^3$.

Ex. 32.

$$x^2-y^2=(x+y)(x-y)$$
$$x^2+2xy+y^2=(x+y)(x+y)$$
$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

The L. C. M. of the $(x+y)$'s is $(x+y)^2$; of the $(x-y)$'s, $x-y$; of the (x^2-xy+y^2)'s, x^2-xy+y^2; of the whole, $(x+y)^2(x-y)(x^2-xy+y^2)$.

L. C. M. — FACTORS BY INSPECTION

Find the L. C. M.:

33. $12x - 36$, $x^2 - 9$, and $x^2 - 5x + 6$

34. $6x^3(a-x)^2$, $8a^2(a-x)^3$, and $12a^2x^2(a-x)^4$

35. $x^2 + x - 6$, $x^2 - 4x + 4$, and $x^3 - 9x$

36. $x^2 - y^2$, $3(x-y)^2$, and $12(x^3 + y^3)$

37. $a^2 + ab + b^2$, $a^3 - b^3$, and $a - b$

NO FACTORS BY INSPECTION

When the expressions cannot be resolved into their prime factors by inspection, it is best to get the H. C. F. of two or more of them and to find the other factors by dividing each expression by the H. C. F.

Find the L. C. M.:

38. $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$

39. $6x^2 - 5ax - 6a^2$ and $4x^3 - 2ax^2 - 9a^3$

40. $4a^2 - 5ab + b^2$ and $3a^3 - 3a^2b + ab^2 - b^3$

41. $2x^3 - 12x^2 + 19x - 12$ and $2x^3 - 6x^2 + 7x - 3$

42. $x^3 - 9x^2 + 23x - 15$ and $x^2 - 8x + 7$

43. $6x^2 + 13x - 28$ and $12x^2 - 31x + 20$

44. $3x^2 - 13x + 12$, $5x^2 - 17x + 6$, and $5x^2 + 3x - 2$.

Ex. 38. The H. C. F. of $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$ is $x - 1$.

Hence,
$$x^3 + x^2 - 2 = (x-1)(x^2 + 2x + 2)$$
$$x^3 + 2x^2 - 3 = (x-1)(x^2 + 3x + 3)$$
$$(x-1)(x^2 + 2x + 2)(x^2 + 3x + 3) = \text{L. C. M.}$$

NOTE. The answers may be expressed as the products of their factors.

SOLUTION OF PROBLEMS

1. To find the L. C. M. of two numbers, divide one of them by their H. C. F. and multiply the quotient by the other. Prove.

2. Solve Ex. 38, p. 74, by the method of Ex. 1.

3. The product of the L. C. M. and H. C. F. of two numbers is the product of the numbers. Prove.

4. The L. C. M. of two numbers is 360 and their H. C. F. is 12. Find the numbers by the use of the principle in Ex. 3.

Let $12x$ and $12y$ equal the numbers, x and y being prime to each other. $144xy = 12 \times 360$; $xy = 30$; *i.e.*, we are to find two numbers prime to each other whose product is 30. They are 1 and 30, 2 and 15, 3 and 10, or 5 and 6. $12x$ and $12y$ are 12 and 360; 24 and 180; 36 and 120; or 60 and 72.

5. The H. C. F. of two or more numbers is the product of all their common prime factors, each taken the greatest number of times it is found in all of them. The L. C. M. of two or more numbers is the product of all their prime factors, each taken the greatest number of times it is found in any one of them. By these principles, solve Ex. 4.

The H. C. F. $12 = 2^2 \cdot 3$; the L. C. M. $360 = 2^3 \cdot 3^2 \cdot 5$. Therefore, 2 must be found not less than twice in each and 3 times in one; 3, not less than once in each and 2 times in one; 5, once in one of them. The possible combinations are:

ONE	THE OTHER	ONE	THE OTHER
$2^3 \cdot 3^2 \cdot 5$	$2^2 \cdot 3$	$2^2 \cdot 3^2 \cdot 5$	$2^3 \cdot 3$
$2^3 \cdot 3 \cdot 5$	$2^2 \cdot 3^2$	$2^2 \cdot 3 \cdot 5$	$2^3 \cdot 3^2$

6. A can walk around a half-mile track in 10 min.; B, in 12 min.; C, in 15 min. If they all start from the same point, at the same time, and in the same direction, how many times must A walk around the track before they are all together at the starting point a second time? Let x = number of times.

7. Solve Ex. 6 by finding the number of minutes before they are all together again.

FRACTIONS

PRINCIPLES

PROPOSITION XXXVIII. THEOREM

Multiplying the numerator multiplies a fraction.

Multiplying the numerator multiplies the number of equal parts that are taken without affecting the size of the parts, and thus multiplies the fraction.

Hence, the principle, Q.E.D.

PROPOSITION XXXIX. THEOREM

Multiplying the denominator divides a fraction.

Multiplying the denominator multiplies the number of equal parts into which the unit is divided, thereby dividing the size of each part without affecting the number of parts taken, and thus divides the fraction.

Hence, the principle, Q.E.D.

PROPOSITION XL. THEOREM

Dividing the numerator divides a fraction.

Dividing the numerator divides the number of equal parts that are taken without affecting the size of the parts, and thus divides the fraction.

Hence, the principle, Q.E.D.

PROPOSITION XLI. THEOREM

Dividing the denominator multiplies a fraction.

Dividing the denominator divides the number of equal parts into which the unit is divided, thereby multiplying the size of each part without affecting the number of parts taken, and thus multiplies the fraction.

Hence, the principle, Q.E.D.

PROPOSITION XLII. THEOREM

Multiplying both numerator and denominator by the same number does not change the value of a fraction.

Since multiplying the numerator multiplies the fraction, and multiplying the denominator divides the fraction, multiplying both terms by the same number first multiplies and then divides the fraction by the same number, and does not, therefore, change the value of the fraction.

Hence, the principle, Q.E.D.

PROPOSITION XLIII. THEOREM

Dividing both numerator and denominator by the same number does not change the value of a fraction.

Since dividing the numerator divides the fraction, and dividing the denominator multiplies the fraction, dividing both terms by the same number first divides and then multiplies the fraction by the same number and does not, therefore, change the value of the fraction.

Hence, the principle, Q.E.D.

NOTE. A general proof may sometimes be given without the representation of numbers by letters.

PRINCIPLES

Proposition XLIV. Theorem

To multiply fractions, multiply the numerators for a new numerator and the denominators for a new denominator.

To prove that $$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

$$\frac{a}{b} \times c = \frac{a \times c}{b}$$

(Multiplying the numerator multiplies the fraction.)

Since c is d times $\frac{c}{d}$, multiplying by c is multiplying by a number d times too large and the result must be d times too large. Since the result is d times too large, it must be divided by d.

$$\frac{a \times c}{b} \div d = \frac{a \times c}{b \times d} \quad \therefore \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

(Multiplying the denominator divides the fraction.)

Hence, the principle, Q.E.D.

Proposition XLV. Theorem

To divide fractions, invert the divisor and proceed as in multiplication.

To prove that $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

$$\frac{a}{b} \div c = \frac{a}{b \times c}$$

(Multiplying the denominator divides the fraction.)

Since c is d times $\frac{c}{d}$, dividing by c is dividing by a number d times too large and the quotient must be d times too small. Since the quotient is d times too small it must be multiplied by d.

$$\frac{a}{b \times c} \times d = \frac{a \times d}{b \times c} \quad \therefore \frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$$

(Multiplying the numerator multiplies the fraction.)

Hence, the principle, Q.E.D.

TO LOWEST TERMS—BY INSPECTION

A fraction is reduced to its lowest terms when both numerator and denominator are prime to each other. The result is obtained by dividing both terms by all their common factors. The subject divides itself into two cases, where common factors can be found by inspection, and where common factors cannot be found by inspection. In the latter case, we find the H. C. F.

Reduce to lowest terms:

1. $\dfrac{18\,x^2y^3z^4}{27\,x^3y^2z^5}$

2. $\dfrac{32\,abc^2}{48\,a^2b^3c^5}$

3. $\dfrac{a^2-2\,ab+b^2}{a^2-b^2}$

4. $\dfrac{a^2-b^2}{a^2+2\,ab+b^2}$

5. $\dfrac{a^2-5\,a+6}{a^2+3\,a-10}$.

6. $\dfrac{a^3-b^3}{a^4+a^2b^2+b^4}$

7. $\dfrac{a^3+b^3}{a^4+a^2b^2+b^4}$

8. $\dfrac{a^3+x^3}{a^2+2\,ax+x^2}$

9. $\dfrac{a^4-b^4}{a^4+2\,a^2b^2+b^4}$

10. $\dfrac{x^6-y^6}{x^6+2\,x^3y^3+y^6}$

11. $\dfrac{x^2-9\,x+20}{x^2-x-12}$

12. $\dfrac{12\,(x^2-y^2)}{18\,(x^2-2\,xy+y^2)}$

13. $\dfrac{21\,(x^3+y^3)}{24\,(x^5+y^5)}$

14. $\dfrac{x^4-y^4}{x^6-y^6}$

15. $\dfrac{x^2-ax-bx+ab}{x^2-ax+bx-ab}$

16. $\dfrac{a^2-b^2+c^2-2\,ac}{a^2+b^2-c^2-2\,ab}$

17. $\dfrac{(a+b)^2-c^2}{(a+b+c)^2}$

18. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$

Ex. 15. $$\frac{x^2-ax-bx+ab}{x^2-ax+bx-ab}=\frac{(x-a)(x-b)}{(x-a)(x+b)}=\frac{x-b}{x+b}$$

TO LOWEST TERMS — NO FACTORS BY INSPECTION

Reduce to lowest terms:

19. $\dfrac{2x^2+19x+35}{3x^2+15x-42}$

20. $\dfrac{x^3-8x^2+19x-12}{x^3-10x^2+29x-20}$

21. $\dfrac{a^3-3a^2+a+2}{2a^3-3a^2-a-2}$

22. $\dfrac{5x^3+2x^2-15x-6}{7x^3-4x^2-21x+12}$

Ex. 19. The H. C. F. is $x+7$. Dividing both terms by $x+7$, $\dfrac{2x+5}{3x-6}$.

TO HIGHER TERMS

As a preparation for addition and subtraction, it is necessary to reduce fractions to equivalent fractions having a common denominator. The result is obtained by multiplying both terms by the same expression.

Reduce to equivalent fractions having their least common denominator:

23. $\dfrac{3}{5a^2b}, \dfrac{2}{3a^3b^3}, \dfrac{1}{2ab}$

24. $\dfrac{5}{3x}, \dfrac{2}{5y^2}, \dfrac{3}{7x^3y}$

25. $\dfrac{1}{x-3}, \dfrac{3}{x^2-9}, \dfrac{5}{3x+9}$

26. $\dfrac{5}{2-2x^2}, \dfrac{1}{1-x^2}, \dfrac{2}{3x-3}$

27. $\dfrac{a-b}{a+b}, \dfrac{a+b}{a^2-b^2}, \dfrac{a^2+b^2}{a-b}$

28. $\dfrac{5}{1+2x}, \dfrac{3x}{1-2x}, \dfrac{4-13x}{1-4x^2}$

29. $\dfrac{1}{(a-b)(b-c)}, \dfrac{1}{(a-b)(a-c)}$

30. $\dfrac{x-2}{x^2+6x+8}, \dfrac{x+2}{x^2+x-12}$

Ex. 23. The least common denominator (L. C. D.) is $30a^3b^3$; L. C. D. $\div$ $5a^2b = 6ab^2$; $\dfrac{3}{5a^2b} = \dfrac{18ab^2}{30a^3b^3}$; and so on.

Ex. 29. The L. C. D. is $(a-b)(b-c)(a-c)$; L. C. D. $\div (a-b)(b-c) =$ $a-c$; $\dfrac{1}{(a-b)(b-c)} = \dfrac{a-c}{(a-b)(b-c)(a-c)}$; and so on.

TO WHOLE OR MIXED NUMBERS

This is the case in division in which the dividend does not exactly contain the divisor. The quotient forms the integral part of the mixed number; the remainder and the divisor form the fractional part.

Reduce to a whole or mixed expression:

31. $\frac{17a^2}{3}$

32. $\frac{25x^2y^2}{6}$

33. $\frac{x^3-y^3}{x+y}$

34. $\frac{x^3+x^2+1}{x^2-2}$

35. $\frac{x^3-x^2}{x^2+1}$

36. $\frac{3x^3-5x^2}{x^2+2}$

Ex. 35. $(x^3-x^2)\div(x^2+1)=x-1+\frac{-x+1}{x^2+1}$, or $x-1-\frac{x-1}{x^2+1}$. The latter is the better form.

WHOLE OR MIXED NUMBERS TO FRACTIONS

This is the case where the quotient, remainder, and divisor are given to find the dividend. We multiply the quotient by the divisor and add the remainder.

Reduce to a single fraction:

37. $5a^2+\frac{2a^2}{3}$

38. $4x^2y^2+\frac{x^2y^2}{6}$

39. $x^2-xy+y^2-\frac{2y^3}{x+y}$

40. $x+1+\frac{2x+3}{x^2-2}$

41. $x-1-\frac{x-1}{x^2+1}$

42. $3x-5-\frac{6x-10}{x^2+2}$

ADDITION AND SUBTRACTION

In addition and subtraction, the addends are reduced to fractions of like denominations and united.

Simplify:

43. $\frac{2x-5}{12}-\frac{3x-11}{18}$

44. $\frac{2a+3}{6}-\frac{3a+5}{8}$

45. $\frac{b-4a}{24a}+\frac{a+5b}{30b}$

46. $\frac{2xy}{15}+\frac{4xy}{5}-\frac{3xy}{7}$

47. $\frac{3z+5y}{6}-\frac{2z+3y}{4}$

48. $\frac{2a+b}{3c}+\frac{5a-4b}{12c}$

49. $\frac{3x+5}{3}+\frac{3x-1}{2}$

50. $\frac{5ax}{6}-\frac{2ax}{9}+\frac{5ax}{12}$

51. $\frac{5x-1}{8}-\frac{3x-2}{7}+\frac{x-5}{4}$

52. $\frac{2x-4y}{5}-\frac{5x+2y-3z}{10}+\frac{x+16y-5z}{15}$

53. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a-4b}{21}$

54. $\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$

Ex. 43.

$$36 = \text{L. C. D.}$$

$$\begin{array}{c}
3(2x-5)-2(3x-11)\,\text{D} \\
6x-15 \\
\underline{-6x+22} \\
\frac{7}{36}
\end{array}$$

12 is contained in the L. C. D. 3 times; 3 times $2x-5$ is $3(2x-5)$; 18 is contained in the L. C. D. 2 times; 2 times $-(3x-11)$ is $-2(3x-11)$.

The denominator belongs under each of these expressions and is indicated by D written at the right.

By simplifying, the new numerator is found to be 7; the L. C. D. is written beneath it.

Simplify:

55. $\frac{a}{a+b}-\frac{a}{a-b}+\frac{3a^2}{a^2-b^2}$

56. $\frac{m+n}{m-n}-\frac{m-n}{m+n}-\frac{4m^2}{m^2-n^2}$

57. $\frac{5}{1+2x}-\frac{3x}{1-2x}-\frac{4-13x}{1-4x^2}$

58. $\frac{x-2a}{x+2a}+\frac{x+2a}{x-2a}+\frac{8ax}{x^2-4a^2}$

59. $\frac{3-x}{1-3x}-\frac{3+x}{1+3x}+\frac{1-16x}{1-9x^2}$

60. $\frac{1}{x^2-y^2}+\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}$

61. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$

62. $\frac{3x^2-8}{x^3-1}-\frac{5x+7}{x^2+x+1}+\frac{2}{x-1}$

63. $\frac{1+2a}{1-2a}-\frac{1-2a}{1+2a}-\frac{8a}{(1-2a)^2}$

Ex. 55.

$$a^2-b^2 = \text{L. C. D.}$$

$$a(a-b)-a(a+b)+3a^2 \quad \text{D}$$

$$\begin{array}{l} a^2-ab \\ -a^2-ab+3a^2 \\ \hline 3a^2-2ab \\ \hline a^2-b^2 \end{array}$$

$a+b$ is contained in the L. C. D. $a-b$ times; $a-b$ times a is $a(a-b)$; $a-b$ is contained in the L. C. D. $a+b$ times; $a+b$ times $-a$ is $-a(a+b)$; a^2-b^2 is contained in the L. C. D. once; once $3a^2$ is $3a^2$.

The denominator belongs under each of these expressions and is indicated by D.

By simplifying, the new numerator is found to be $3a^2-2ab$.

ADDITION AND SUBTRACTION

Simplify:

64. $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-c)(b-a)}+\dfrac{a+b}{(c-a)(c-b)}$

65. $\dfrac{b+c}{(a-b)(a-c)}-\dfrac{a+c}{(b-c)(a-b)}+\dfrac{a+b}{(a-c)(b-c)}$

66. $\dfrac{2x+y}{x+y}-1-\dfrac{y}{y-x}+\dfrac{x^2}{y^2-x^2}$

67. $\dfrac{x+2}{x^3-x}-\dfrac{1}{2x+2}-\dfrac{3}{2x-2}+\dfrac{2}{x}$

Ex. 64. $(a-b)(a-c)(b-c)=\text{L. C. D.}$

$$(b+c)(b-c)-(a+c)(a-c)+(a+b)(a-b)\ \text{D}$$

$$\begin{array}{l} b^2-c^2 \\ \quad +c^2-a^2 \\ \underline{-b^2 \qquad +a^2} \end{array}$$

$$\frac{0}{(a-b)(a-c)(b-c)}=0$$

The L. C. D. must contain $(a-b)(a-c)$; it must contain $b-c$ and $b-a$; we retain $b-c$, but not $b-a$, because $b-a$ is contained in $a-b$; the L. C. D. must contain $c-a$ and $c-b$; we do not retain them, because $c-a$ is contained in $a-c$; and $c-b$, in $b-c$.

$(a-b)(a-c)$ is contained in the L. C. D., $b-c$ times; $b-c$ times $b+c$ is $(b+c)(b-c)$.

$(b-c)(b-a)$ is contained in the L. C. D. $-(a-c)$ times; $-(a-c)$ times $(a+c)$ is $-(a+c)(a-c)$.

$(c-a)(c-b)$ is contained in the L. C. D. $--(a-b)$ times; $--(a-b)$ times $a+b$ is $+(a+b)(a-b)$.

By simplifying, the numerator becomes 0; hence, the result is 0.

NOTE. As a preparation for the solution of such examples, they may be written in symmetrical form. Thus, the denominator of the first fraction is arranged symmetrically already; in the denominator of the second, $b-a$ occurs instead of the symmetrical form, $a-b$. Both terms of the second fraction may be multiplied by -1, making $\dfrac{-a-c}{(b-c)(a-b)}$, or $-\dfrac{a+c}{(b-c)(a-b)}$; and so on.

MULTIPLICATION AND DIVISION

Simplify:

68. $\dfrac{x^2+5xy+6y^2}{x^2-4xy-21y^2}\times\dfrac{x^2-7xy}{x^2-4y^2}$

69. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}$

70. $\dfrac{x^3+y^3}{x^2-2xy+y^2}\div\dfrac{x^2+xy}{x-y}$

71. $\dfrac{x-1-\dfrac{12}{x+3}}{x-5+\dfrac{12}{x+3}}$

72. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$

73. $\dfrac{\dfrac{1}{1-a}-\dfrac{1}{1+a}}{\dfrac{1}{1-a}+\dfrac{1}{1+a}}$

74. $\dfrac{1+\dfrac{(a-b)^2}{4ab}}{1+\dfrac{b^2+a^2}{2ab}}$

75. $\left(\dfrac{a^3-b^3}{a^3+b^3}\times\dfrac{(a+b)^2}{a-b}\right)\div\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$

Ex. 68. $\dfrac{(x+3y)(x+2y)}{(x-7y)(x+3y)}\times\dfrac{x(x-7y)}{(x+2y)(x-2y)}=\dfrac{x}{x-2y}$

Ex. 71. $(\ )=\dfrac{(x+3)(x-1)-12}{(x-5)(x+3)+12}=\dfrac{x^2+2x-15}{x^2-2x-3}=\dfrac{x+5}{x+1}$

We multiply both numerator and denominator by the least common multiple of the denominators of both terms, or by $x+3$. We could have reduced each term to a fraction, inverted the divisor, and proceeded as in multiplication.

Ex. 72. $(\ )=\dfrac{(a^2+b^2)^2-(a^2-b^2)^2}{a^4-b^4}\div\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}$

$$=\frac{4a^2b^2}{a^4-b^4}\times\frac{a^2-b^2}{4ab}=\frac{ab}{a^2+b^2}$$

When the least common multiple of the denominators divided by any denominator gives a long quotient, it is best to proceed in this way.

MISCELLANEOUS

Simplify:

76. $\dfrac{x^2-14x+24}{y^2+9y-36} \div \left(\dfrac{x^2+4x-12}{y^2+2y-15} \times \dfrac{xy+5x}{xy+6y}\right).$

77. $\dfrac{6a^3+27a^2+21a-9}{6a^3+10a^2-30a+8}$

78. $\dfrac{x^2-xy+xz-yz}{x^2+xy+xz+yz}$

79. $\left(x-\dfrac{1}{x}\right)-\left(y-\dfrac{1}{x}\right)$

80. $\dfrac{\dfrac{m-n}{m+n}+\dfrac{m+n}{m-n}}{\dfrac{m-n}{m+n}-\dfrac{m+n}{m-n}}$

81. $\dfrac{x^2+5x+6}{x^2+7x+12}$

82. $\dfrac{(x^3-a^3)(x^3+a^3)}{x^2-a^2}$

83. $\dfrac{4x^2-9y^2}{(2x-3y)^2}$

84. $\left(4m-\dfrac{2n}{x}\right)+\left(m-\dfrac{n}{x}\right)$

85. $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{4ax}{a^2-x^2}}$

86. $\dfrac{x^2-9x+20}{x^2-2x-15}$

87. $\dfrac{a-4b}{4a}-\dfrac{3a+5b}{6a}+\dfrac{2a-3b}{8a}-\dfrac{3a-7b}{3a}$

88. $\dfrac{4a^2}{a^4+a^2+1}-\dfrac{2}{a^2-a+1}+\dfrac{2a}{a^2+a+1}$

89. $\dfrac{2x-6}{x^2+3x+2}-\dfrac{x+2}{x^2-2x-3}-\dfrac{x+1}{x^2-x-6}$

90. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$

91. $\dfrac{m^2-bx}{(m+b)(m+x)}-\dfrac{mx-b^2}{(b+x)(b+m)}-\dfrac{mb-x^2}{(x+m)(x+b)}$

SOLUTION OF PROBLEMS

Multiplying or dividing both members of an equation by the same number cannot affect the equality.

Thus, $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$; multiplying both members by 6, $5 = 3 + 2$.

1. One half of A's money plus $\frac{1}{3}$ of his money is $100. State the relation.

2. Solve the equation of Ex. 1 by uniting the fractions. Solve by clearing of fractions; *i.e.*, by multiplying both members by 6, the least common denominator. Which method do you prefer?

Uniting First

$$\begin{aligned} \frac{x}{2} + \frac{x}{3} &= 100 \\ \frac{5}{6}x &= 100 \\ x &= 100 \div \frac{5}{6} \\ &= 120 \end{aligned}$$

Clearing First

$$\begin{aligned} \frac{x}{2} + \frac{x}{3} &= 100 \\ 3x + 2x &= 600 \\ 5x &= 600 \\ x &= 120 \end{aligned}$$

3. Find how much A had in Ex. 1 by the indirect method. Do you prefer the direct or the indirect?

One half of A's money plus $\frac{1}{3}$ of his money is $\frac{5}{6}$ of his money; since $\frac{5}{6}$ of his money is $100, $\frac{1}{6}$ of his money is $\frac{1}{5}$ of $100, or $20; $\frac{6}{6}$, or the whole, is 6 times $20, or $120.

4. Find the number whose third and fourth parts together equal 14.

5. Find the number whose third part exceeds its fourth part by 5.

6. A stake is $\frac{1}{5}$ in the mud, $\frac{2}{7}$ in the water, and 18 feet above the water. What is the length of the stake?

7. One third of a number added to 3 times the number is equal to 50. What is the number?

8. Three fifths of a certain number exceeds $\frac{1}{3}$ of it by 7. What is the number?

SIMPLE PROBLEMS

9. Divide 207 into two parts such that $\frac{1}{4}$ the greater shall exceed $\frac{2}{7}$ the less by 3.

Let $x =$ the greater, 116

then $207 - x =$ the less, 91

$$\frac{x}{4} = \frac{1}{4}\text{, the greater}$$

$$\frac{2}{7}(207 - x) = \frac{2}{7}\text{, the less}$$

$$\therefore \frac{x}{4} - \frac{2}{7}(207 - x) = 3 \quad (1)$$

$$7x - 8(207 - x) = 84$$

$$x = 116$$

PROOF

1. $116 + 91 = 207$
2. $\frac{1}{4}$ of $116 - \frac{2}{7}$ of $91 = 3$

EXPLANATION. Let x equal the greater; then $207 - x$ must equal the less, because their sum is 207; $\frac{x}{4} - \frac{2}{7}(207 - x) = 3$, because $\frac{1}{4}$ of the greater exceeds $\frac{2}{7}$ the less by 3; whence $x = 116$, the greater; and $207 - x = 91$, the less.

PROOF. The first condition is that their sum is 207; the sum of 116 and 91 is 207. The second condition is that $\frac{1}{4}$ of the greater exceeds $\frac{2}{7}$ the less by 3; $\frac{1}{4}$ of 116 exceeds $\frac{2}{7}$ of 91 by 3.

NOTE. We multiply both members of (1) by 28, the least common denominator.

10. Solve Ex. 9 by letting x equal the less.

11. A man spends $\frac{1}{5}$ of his yearly income for house rent, $\frac{1}{2}$ of the remainder for provisions, $\frac{2}{5}$ of the remainder for miscellaneous expenses, and lays up \$240. Find his income.

12. A son's age is $\frac{2}{5}$ of his father's age, but in 16 years he will be $\frac{4}{7}$ as old as his father. Find the present age of each.

13. What is the time of day if $\frac{2}{3}$ of the time before noon equals the time since midnight?

14. A spent $\frac{1}{5}$ of his money, then received \$8; after spending $\frac{1}{3}$ of the sum, he had \$40 remaining. How much had he at first?

15. B is 72 years old, and $\frac{2}{3}$ of his age is equal to C's age. How long is it since B was 5 times as old as C?

SIMPLE PROBLEMS

16. In a mixture of wine and water the wine was 25 gallons more than $\frac{1}{2}$ of the mixture, and the water 5 gallons less than $\frac{1}{3}$ of the mixture. How many gallons were there of each? Let $x =$ gal. in mixture.

17. Twelve men engaged a supper for \$20; but before paying the bill a number of them withdrew, by which each person's bill was increased \$$\frac{5}{6}$. How many withdrew?

18. Out of a full cask of wine, $\frac{1}{5}$ part had leaked away; afterwards 10 gallons were drawn out, when the cask was found to be $\frac{2}{3}$ full. How much did it hold?

19. Eight persons hired a coach for \$48; but before starting they took in an additional number, by which each person's bill was diminished \$2. How many were added?

20. A's age 10 years ago was $\frac{1}{2}$ of what his age will be 10 years hence. Find his present age.

21. A man bought a cow, a colt, and a horse. The cow cost \$40; the colt, as much as the cow and $\frac{1}{2}$ as much as the horse; and the horse, $\frac{3}{2}$ as much as the colt and cow together. What was the cost of each?

22. Find the number the sum of whose $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ exceeds the sum of its $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ by 482.

23. If a certain number is increased by 2, the sum multiplied by 3, the product diminished by 4, and the remainder divided by 5, the result is 4. What is the number? Let $x =$ the number.

24. Solve the example indirectly. It will be necessary to begin at the end and to proceed backwards. If the result is 4 after the division by 5, before such division the number was 5×4, or 20; and so on.

25. State clearly the difference between a direct or algebraic solution, and an indirect solution or solution by analysis.

PROBLEMS — BUYING AND SELLING

26. A boy bought apples at 3 for a cent and the same number at 2 for a cent; he sold them at 5 for 2 cents and thereby lost 10 ¢. How many of each kind did he purchase? Solve both directly and indirectly.

$$\text{Let } x = \text{the number of each kind, 300}$$

$$\frac{x}{3} = \text{cost of one kind in cents}$$

$$\frac{x}{2} = \text{cost of other kind in cents}$$

$$2 \times \frac{2x}{5} = \text{selling price of both kinds in cents}$$

$$\frac{5x}{6} - \frac{4x}{5} = 10$$

$$x = 300$$

Note. The indirect solution is quite simple after the plan of procedure has been discovered, but more skill is required to form the plan.

27. A woman sold eggs and apples; the eggs were worth 5 cents a dozen more than the apples, and 8 dozen eggs were worth as much as $13\frac{5}{7}$ dozen apples. What was the price of each per dozen?

28. A girl bought one dollar's worth of postage stamps, receiving a certain number of five-cent stamps, 3 less than 2 times as many two-cent stamps, and 2 less than 3 times as many one-cent stamps. How many stamps of each kind did she buy?

29. If I gain 1 ¢ apiece by selling oranges at 60 ¢ a dozen, how much apiece do I gain by selling them at 54 ¢ a dozen?

30. By selling oranges at 5 ¢ each, I shall lose 12 ¢; by selling them at 8 ¢ each, I shall gain 24 ¢. How many oranges have I?

31. I sell 12 melons for a certain price; had I sold 4 more for the same money, the price of each melon would have been diminished 4 ¢. For how much did I sell each melon?

32. A boy bought apples at a for a cent, sold them at b for a cent, and gained c cents. How many apples were there?

PROBLEMS — PERCENTAGE

33. A farmer had 150 sheep and sold 60 of them. What per cent of the whole did he sell?

34. A farmer sold 60 sheep, or 40% of his flock. How many did he have at first? Let $x =$ number at first.

Ex. 33

Let $x =$ per cent, 40

$$\frac{x}{100} \times 150 = \text{number sold}$$
$$60 = \text{number sold}$$
$$\therefore \frac{x}{100} \times 150 = 60$$
$$150x = 6000$$
$$x = 40$$

Ex. 34

Let $x =$ number at first, 150

$$.40x = \text{number sold}$$
$$60 = \text{number sold}$$
$$\therefore .40x = 60$$
$$x = 60 \div .40$$
$$x = 150$$

35. A dealer bought a horse for \$90 and sold him for \$99. What per cent did he gain?

36. A man bought a farm for \$2000. At what per cent above cost must he sell it to gain \$400?

37. An agent's commission from the sale of goods was \$200 at 2%. What was the selling price?

38. In percentage the base is B; the rate, R; and the percentage P. Find R in terms of P and B. Solve Ex. 33 by substituting in this formula.

39. Find B in terms of P and R. Solve Ex. 34 by substituting in this formula.

40. What per cent of my investment will I realize by buying 6% stock at 115? Let $x =$ the per cent.

41. Suppose 10% state stock is 20% better in market than 4% railroad stock; and A's income is \$500 from each. How much money had he paid for each, the whole investment bringing $6\frac{2}{333}\%$?

42. A farmer had a sheep and sold b of them. What per cent of the whole remained?

PROBLEMS — PERCENTAGE

43. By selling a horse for $99 a dealer lost 10%. How much did the horse cost?

44. By selling a horse for $99 a dealer gained 10%. How much did the horse cost?

Ex. 43

Let x = cost in $, 110

$$\begin{aligned} .10\,x &= \text{loss in \$} \\ .90\,x &= \text{selling price in \$} \\ 99 &= \text{selling price in \$} \\ \therefore\ .90\,x &= 99 \\ x &= 110 \end{aligned}$$

Ex. 44

Let x = cost in $, 90

$$\begin{aligned} .10\,x &= \text{gain in \$} \\ 1.10\,x &= \text{selling price in \$} \\ 99 &= \text{selling price in \$} \\ \therefore\ 1.10\,x &= 99 \\ x &= 90 \end{aligned}$$

45. By selling silk for $330 I gained 10%. What was the cost?

46. By selling silk for $270 I lost 10%. What was the cost?

47. A merchant marked goods at 20% above cost and sold them at this price for $240. What was the cost?

48. A merchant marked goods at 20% above cost and sold them at a reduction of 10% from the marked price for $1080. How much did he gain?

49. An agent's commission at 2% for purchasing goods, plus their first cost, is $1020. What was the first cost?

50. By selling goods for $375 a merchant gained 25%. What would have been the selling price if the gain had been 20%?

51. By selling goods for $375 a merchant lost 25%. What would have been the selling price if the loss had been 20%?

52. A man sold an article at 20% loss; had it cost $300 less, he would have gained 20%. What was the cost?

53. A quantity of sugar was sold at 10% gain; if it had cost $120 more, the same selling price would have entailed a loss of 10%. Find the cost of the sugar.

PROBLEMS — SIMPLE INTEREST

54. At what rate will $50 amount to $65 in 5 yr. at simple interest?

Let x = number of per cent, 6

$$50 \times 5 \times \frac{x}{100} = \frac{5x}{2} = \text{interest in dollars}$$

$$50 + \frac{5x}{2} = 65$$

$$x = 6$$

Explanation. Let x equal the number of per cent; the gain in dollars of $50 in 5 yr. at x per cent is $50 \times 5 \times \frac{x}{100}$, or $\frac{5x}{2}$; the amount in dollars is $50 + \frac{5x}{2}$; $\therefore\ 50 + \frac{5x}{2} = 65$; whence $x = 6$, the number of per cent.

Proof. The condition is that the $50 shall amount to $65 in 5 yr. In 5 yr. at 6%, $50 amounts to $65.

55. At what per cent will $100 gain $18 in 3 yr.? Let x = the per cent.

56. In what time will $125 gain $22.50 at 6%? Let x = the number of years.

57. Find the time in which $340 will produce $13.60 interest at 5%.

58. What principal will gain $22.50 in 3 yr. at 6%? Let x = the number of dollars.

59. At what per cent will a sum double in 10 yr.?

60. At what per cent will $260 amount to $289.90 in 2 yr. 3 mo. 18 da.?

61. What principal will amount to $289.90 in 2 yr. 3 mo. 18 da. at 5%?

62. In what time will $260 amount to $289.90 at 5%?

63. At what per cent will P dollars gain 1 dollar in t years?

64. In what time will P dollars amount to A dollars at r%?

65. At what rate will P dollars amount to A dollars in t years?

EQUATIONS OF THE FIRST DEGREE

ONE UNKNOWN QUANTITY

An equation containing one unknown quantity is of the first degree if the only exponent of the unknown quantity is one. To find the value of the unknown quantity, it is necessary to reduce the equation to the general form, $ax = b$, and to divide both members by a. Thus:

First Degree	Form $ax = b$	Divided by a
1. $3x + 2 = 20$,	$3x = 18$,	$x = 6$
2. $3x - 4 = x + 6$,	$2x = 10$,	$x = 5$

The principles which govern these operations have been illustrated and practiced in the solution of preceding problems, but it remains to prove them and to apply them to complex expressions.

1. To reduce to the form, $ax = b$, it may be necessary to transpose the terms containing the unknown quantity to the left-hand member and all other terms to the right-hand member.

Proposition XLVI. Theorem

To transpose a term from one member of an equation to the other, change its sign.

Let us take $\quad ax - b = 0$

To prove that $\quad ax = b$

Adding the same quantity to both members of an equation cannot affect the equality. By adding b to both members of $ax - b = 0$, we obtain $ax = b$.

Hence, the principle, Q.E.D.

TRANSPOSING

Find the value of x:

1. $ax - a = cx - c$
2. $ax + bx = cx - d$
3. $ax + c = bx - 2c$
4. $2x + bx - a = 3x - 2c$
5. $ax - c + d = bx + 2d$
6. $5bc - 8ax = 10ax - 13bc$

Ex. 5. $$ax - c + d = bx + 2d$$

Transposing, $$ax - bx = c - d + 2d$$

Uniting, $$(a - b)x = c + d$$

Dividing by $a - b$, $$x = \frac{c + d}{a - b}$$

Find the value of x:

7. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=15$
8. $8(x-3)-6(x-7)=2(x+2)+2(x-5)$
9. $n = bx - a(c - x)$.
10. $m = a(x - b) + cx$
11. $(x-1)(x-2)(x+4)=(x+2)(x+3)(x-4)$
12. $14 - x - 5(x-3)(x+2)+(5-x)(4-5x)=45x-76$

Ex. 7. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=15$

$$\left.\begin{array}{r} 7x - x^2 \\ 2x \\ 3x + x^2 \\ -5x \\ -2x \end{array}\right\} = \left\{\begin{array}{r} 14 \\ 15 \\ -\ 2 \\ -12 \\ 15 \end{array}\right.$$

$$5x = 30$$

$$x = 6$$

It is well to transpose each term at the instant of multiplication. See p. 34. $x \times 7 = 7x$; $x \times -x = -x^2$; $-2 \times 7 = -14$, transposing, $+14$; $-2 \times -x = 2x$; $x \times x = x^2$; $x \times 3 = 3x$; $-5 \times x = -5x$; $-5 \times 3 = -15$, transposing, $+15$; and so on.

CLEARING OF FRACTIONS

2. To reduce to the form $ax = b$, it may be necessary to clear the equation of fractions.

PROPOSITION XLVII. THEOREM

To clear an equation of fractions, multiply both members by the least common multiple of the denominators.

Multiplying both members of an equation by the same number cannot affect the equality.

Multiplying each term by the least common multiple of the denominators must give an integer for each product, and must, therefore, clear the equation of fractions.

Hence, the principle, Q.E.D.

Find the value of x:

13. $\dfrac{3x}{2}+\dfrac{x-3}{3}=\dfrac{3x}{4}-\dfrac{3-5x}{6}$

14. $\dfrac{2x}{3}-\dfrac{3x}{4}=-2$

15. $\dfrac{3x-7}{4}-\dfrac{3x-5}{3}=\dfrac{5}{12}$

16. $\dfrac{2x-3}{5}-\dfrac{3x-7}{4}=-\dfrac{3}{5}$

17. $\dfrac{2x-1}{5}+\dfrac{6x-4}{7}=\dfrac{7x+12}{11}$

18. $\dfrac{2x}{5}-\dfrac{22}{3}=\dfrac{x}{6}-\dfrac{x}{2}$

19. $\dfrac{3x}{7}-\dfrac{x}{5}=\dfrac{2x}{3}+\dfrac{46}{21}$

20. $\dfrac{x-1}{3}-\dfrac{x-2}{4}=\dfrac{x-3}{5}+\dfrac{3}{10}$

21. $\dfrac{x^2-1}{3}-\dfrac{x^2+1}{8}=\dfrac{5x^2-x}{24}$

22. $\dfrac{3x-4}{2}=\dfrac{6x-5}{8}+\dfrac{3x-1}{16}$

Ex. 13. $$\frac{3x}{2}+\frac{x-3}{3}=\frac{3x}{4}-\frac{3-5x}{6}$$

Multiplying by 12, $$18x+4(x-3)=9x-2(3-5x)$$

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$$x=2$$

NOTE. In multiplying $\frac{x-3}{3}$ by 12, it is better to indicate the result as $4(x-3)$ than to perform the multiplication in full, because there is less liability to err.

Find the value of x:

23. $\dfrac{3}{1-x}-\dfrac{2}{1+x}+\dfrac{1}{x^2-1}=0$

24. $\dfrac{m}{x-a}=\dfrac{n}{x-b}$

25. $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}$

26. $\dfrac{4}{x^2-1}+\dfrac{1}{x+1}=\dfrac{1}{1-x}$

27. $\dfrac{2x+1}{5}-\dfrac{3x-2}{6x+3}=\dfrac{6x-1}{15}$

28. $\dfrac{11}{6x+12}-\dfrac{7}{8x-10}=\dfrac{-17}{6(x+2)}$

29. $\dfrac{7}{8}+\dfrac{3}{3-x}=\dfrac{5}{21}$

30. $\dfrac{6}{x+5}=\dfrac{x+1}{4}-\dfrac{x+3}{4}$

Ex. 23. $$\frac{3}{1-x}-\frac{2}{1+x}+\frac{1}{x^2-1}=0$$

$$(1-x)(1+x)=\text{L. C. D.}$$

Mul. by L. C. D., $$3(1+x)-2(1-x)-1=0$$

$$x=0$$

NOTE. $$\frac{3}{1-x}\times(1-x^2)=3(1+x)\cdots\frac{1}{x^2-1}\times(1-x^2)=-1$$

The device illustrated in Ex. 64, p. 84, of arranging the terms in symmetrical form, is often employed as a preparation for clearing of fractions. Thus,

Ex. 23. Multiplying both terms of $\dfrac{1}{x^2-1}$ by -1, we obtain $\dfrac{-1}{1-x^2}$, or $-\dfrac{1}{1-x^2}$, and we may write the equation,

$$\frac{3}{1-x}-\frac{2}{1+x}-\frac{1}{1-x^2}=0$$

Clearing of fractions, we obtain the same result as above. It is simpler to multiply $-\dfrac{1}{1-x^2}$ than $\dfrac{1}{x^2-1}$ by $1-x^2$, but this gain does not offset the labor of arranging the terms.

CLEARING OF FRACTIONS

2. To reduce to the form $ax = b$, it may be necessary to clear the equation of fractions.

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To clear an equation of fractions, multiply both members by the least common multiple of the denominators.

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Multiplying each term by the least common multiple of the denominators must give an integer for each product, and must, therefore, clear the equation of fractions.

Hence, the principle, Q.E.D.

Find the value of x:

13. $\frac{3x}{2} + \frac{x-3}{3} = \frac{3x}{4} - \frac{3-5x}{6}$

14. $\frac{2x}{3} - \frac{3x}{4} = -2$

15. $\frac{3x-7}{4} - \frac{3x-5}{3} = \frac{5}{12}$

16. $\frac{2x-3}{5} - \frac{3x-7}{4} = -\frac{3}{5}$

17. $\frac{2x-1}{5} + \frac{6x-4}{7} = \frac{7x+12}{11}$

18. $\frac{2x}{5} - \frac{22}{3} = \frac{x}{6} - \frac{x}{2}$

19. $\frac{3x}{7} - \frac{x}{5} = \frac{2x}{3} + \frac{46}{21}$

20. $\frac{x-1}{3} - \frac{x-2}{4} = \frac{x-3}{5} + \frac{3}{10}$

21. $\frac{x^2-1}{3} - \frac{x^2+1}{8} = \frac{5x^2-x}{24}$

22. $\frac{3x-4}{2} = \frac{6x-5}{8} + \frac{3x-1}{16}$

Ex. 13. $$\frac{3x}{2} + \frac{x-3}{3} = \frac{3x}{4} - \frac{3-5x}{6}$$

Multiplying by 12, $$18x + 4(x-3) = 9x - 2(3-5x)$$

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$$x = 2$$

NOTE. In multiplying $\frac{x-3}{3}$ by 12, it is better to indicate the result as $4(x-3)$ than to perform the multiplication in full, because there is less liability to err.

Find the value of x:

23. $\dfrac{3}{1-x}-\dfrac{2}{1+x}+\dfrac{1}{x^2-1}=0$

24. $\dfrac{m}{x-a}=\dfrac{n}{x-b}$

25. $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}$

26. $\dfrac{4}{x^2-1}+\dfrac{1}{x+1}=\dfrac{1}{1-x}$

27. $\dfrac{2x+1}{5}-\dfrac{3x-2}{6x+3}=\dfrac{6x-1}{15}$

28. $\dfrac{11}{6x+12}-\dfrac{7}{8x-10}=\dfrac{-17}{6(x+2)}$

29. $\dfrac{7}{8}+\dfrac{3}{3-x}=\dfrac{5}{21}$

30. $\dfrac{6}{x+5}=\dfrac{x+1}{4}-\dfrac{x+3}{4}$

Ex. 23.
$$\frac{3}{1-x}-\frac{2}{1+x}+\frac{1}{x^2-1}=0$$
$$(1-x)(1+x)=\text{L. C. D.}$$

Mul. by L. C. D., $$3(1+x)-2(1-x)-1=0$$
$$x=0$$

NOTE. $\dfrac{3}{1-x}\times(1-x^2)=3(1+x)\cdots\dfrac{1}{x^2-1}\times(1-x^2)=-1$

The device illustrated in Ex. 64, p. 84, of arranging the terms in symmetrical form, is often employed as a preparation for clearing of fractions. Thus,

Ex. 23. Multiplying both terms of $\dfrac{1}{x^2-1}$ by -1, we obtain $\dfrac{-1}{1-x^2}$, or $-\dfrac{1}{1-x^2}$, and we may write the equation,

$$\frac{3}{1-x}-\frac{2}{1+x}-\frac{1}{1-x^2}=0$$

Clearing of fractions, we obtain the same result as above. It is simpler to multiply $-\dfrac{1}{1-x^2}$ than $\dfrac{1}{x^2-1}$ by $1-x^2$, but this gain does not offset the labor of arranging the terms.

CLEARING OF FRACTIONS

A. Before clearing of fractions, it is often of advantage to unite the terms of each member separately. Denominators somewhat alike should be transposed to the same member before the terms are united. See Exs. 33 and 35.

Find the value of x:

31. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}$

32. $\dfrac{1}{x-1}-\dfrac{1}{x-2}=\dfrac{1}{x-3}-\dfrac{1}{x-4}$

33. $\dfrac{2x-5}{5}+\dfrac{x-3}{2x-15}=\dfrac{4x-3}{10}-\dfrac{11}{10}$

34. $\dfrac{4}{x-7}+\dfrac{1}{x-9}=\dfrac{1}{x-5}+\dfrac{4}{x-8}$

35. $\dfrac{6x+7}{15}-\dfrac{2x-2}{7x-6}=\dfrac{2x+1}{5}$

36. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$

Ex. 31.

$(x-3)(x-1)-(x-2)^2\ \text{D} \qquad (x-5)(x-3)-(x-4)^2\ \text{D}$

$$\begin{array}{r} x^2-4x+3 \\ \underline{-x^2+4x-4} \end{array} \qquad \begin{array}{r} x^2-8x+15 \\ \underline{-x^2+8x-16} \end{array}$$

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-4)(x-5)}$$

$$\frac{1}{(x-2)(x-3)}=\frac{1}{(x-4)(x-5)} \qquad (1)$$

Uniting the left-hand member, we obtain $\dfrac{-1}{(x-2)(x-3)}$; the right-hand member, $\dfrac{-1}{(x-4)(x-5)}$. Multiplying both members by -1, we obtain (1). Clearing of fractions and reducing, we find that $x=3\frac{1}{2}$.

B. Before clearing of fractions, it is often wise to reduce the fractions to mixed numbers. To discover whether this device is of value, the example should be inspected to see if the integral parts of the mixed numbers will cancel.

Find the value of x:

37. $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\frac{1}{6}$

38. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x$

39. $\frac{ax+b}{ax-b}-\frac{3b}{ax+b}=\frac{a^2x^2+b^2}{a^2x^2-b^2}$

40. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}$

41. $\frac{2x}{2x+1}+\frac{5}{2x-1}+\frac{2x-5}{2x+1}=2$

42. $\frac{x+2}{x}+\frac{x-7}{x-5}-\frac{x+3}{x+1}=\frac{x-6}{x-4}$

43. $\frac{x-1}{x-2}+\frac{x+1}{x+2}=\frac{2(x^2+4x+1)}{(x+2)^2}$

44. $\frac{x}{x-3}+\frac{x-1}{x-5}=\frac{2x^2-5x-15}{x^2-8x+15}$

45. $\frac{4x^3+4x^2+8x+5}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$

Ex. 37.
$$1+\frac{x}{2x-1}-1-\frac{x}{3x-2}=\frac{1}{6}$$
$$\frac{x}{2x-1}-\frac{x}{3x-2}=\frac{1}{6}$$

By inspection, we discover that the integral parts of the mixed numbers (1 and -1) will cancel.

$\frac{4x-2}{3x-2}=1+\frac{x}{3x-2}$; the minus sign before the original fraction changes the sign of each part of the mixed number.

SOLUTION OF PROBLEMS

1. A man was hired 50 days; for every day he worked he was to receive 75 cents, and for every day he was idle he was to pay 25 cents for his board; at the expiration of the time he received \$27.50. How many days did he work?

$$\begin{aligned} \text{Let } x &= \text{no. days he worked, } 40 \\ 50 - x &= \text{no. days he was idle, } 10 \\ 75x &= \text{cents received for labor} \\ 25(50 - x) &= \text{cents paid for board} \\ 75x - 25(50 - x) &= 2750 \\ x &= 40 \end{aligned}$$

PROOF

$40 \times \$.75 - 10 \times \$.25 = \$27.50.$

EXPLANATION. Let x equal number days he worked; then $50 - x$ must equal the number of days he was idle, because he agreed to work 50 days; $75x$ equals the number of cents he received for his labor, and $25(50 - x)$ the number of cents he paid for board; $75x - 25(50 - x) = 2750$, because he received in all 2750 cents; whence $x = 40$.

PROOF. The condition is that for the whole time he received \$27.50; $40 \times \$.75 - 10 \times \$.25 = \$27.50$.

2. A man engaged to work for 30 days on the condition that he was to receive \$1.50 for each day he worked, and forfeit 50 cents for each day he was idle; at the end of the 30 days he received \$27. How many days had he worked?

3. A carpenter received \$3.50 a day for his labor, and paid \$1.00 a day for his board; at the end of 24 days he received \$39. How many days did he work?

4. A laborer agreed to serve for a days on condition that for every day he worked he should receive b dollars and for every day he was idle he should forfeit c dollars; at the end of the time he received d dollars. How many days did he work, and how many was he idle?

NOTE. The pupil should try the indirect solution of these problems. It is simpler than the direct.

5. A can do a piece of work in 10 days, B in 12 days; with the help of C they can do it in 4 days. In what time can C do it alone?

Let $x =$ number of days required by C, 15

$\frac{1}{10}, \frac{1}{12}, \frac{1}{x} =$ the parts A, B, C, can do in 1 day

$\frac{1}{4} =$ part all can do in 1 day

$\frac{1}{10}+\frac{1}{12}+\frac{1}{x} =$ part all can do in 1 day

$$\frac{1}{10}+\frac{1}{12}+\frac{1}{x}=\frac{1}{4}$$

$$x = 15$$

PROOF

1. $\frac{1}{10}+\frac{1}{12}+\frac{1}{15}=\frac{1}{4}$

EXPLANATION. Let x equal the number of days required by C; then $\frac{1}{10}, \frac{1}{12}, \frac{1}{x}$ equal the parts each can do in 1 day respectively, because in 1 day each can do $\frac{1}{n}$ of what he can do in n days. And so on.

PROOF. The condition is that if A can do a piece of work in 10 days, and B in 12 days, A, B, and C can together do it in 4 days. The sum of $\frac{1}{10}$, $\frac{1}{12}$, and $\frac{1}{15}$, the parts that they do in 1 day, is equal to $\frac{1}{4}$, the part that they do together in 1 day.

6. A cistern can be filled in 4 hours by 2 pipes running together, and in $6\frac{1}{4}$ hours by one pipe alone. In how many hours can the other pipe alone fill it?

7. A tank is filled by a pipe in 30 minutes; it is emptied by another pipe in 50 minutes. In what time will it be filled with both pipes running at once?

8. A can reap a field in a days; B in b days; and C in c days. In what time can they reap it together?

9. A cistern is filled by one pipe in 8 hours, by another in 9 hours. How soon will the cistern be filled by both pipes running together?

10. Two men working separately can do a piece of work in 15 days and 16 days respectively; with the aid of a third they can do it in 6 days. How long would it take the third man to do it?

SOLUTION OF PROBLEMS — COMMON EXPERIENCE

11. A crew can row downstream 8 miles an hour and upstream 6 miles an hour. How far downstream can they row and return in 7 hours?

Let $x =$ number of miles

then $\frac{x}{8} =$ number of hours to row down

and $\frac{x}{6} =$ number of hours to row up

$$\therefore \frac{x}{8} + \frac{x}{6} = 7$$

$$x = 24$$

Explanation. Let x equal the number of miles; then $\frac{x}{8}$ must equal the number of hours to row downstream, because the crew rows 8 miles per hour downstream; $\frac{x}{6}$ must equal the number of miles to row upstream, etc.

Proof. The condition is, etc.

12. A man who can row 4 miles an hour in still water rows uniformly down a stream one hour; then he floats with the current half an hour, and rows back in $3\frac{1}{2}$ hours. How rapid is the current?

13. How far may a person ride in a car going at the rate of 30 miles an hour if he returns at the rate of 18 miles an hour and is gone but 10 hours?

14. A steamer whose rate of sailing in still water is 15 miles an hour moves down a stream whose current is 3 miles an hour, and returns, making a round trip in 5 hours. How far did she go downstream?

15. A person has just 5 hours at his disposal; how far can he ride in a buggy, going 10 miles an hour, and walk back at the rate of 4 miles an hour?

16. A man rows downstream at the rate of a miles an hour and returns at the rate of b miles an hour. How far downstream can he go and return in c hours?

17. A hare takes 4 leaps while a greyhound takes 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. If the hare has a start of 50 of her own leaps, how many leaps must the greyhound make to overtake her?

Let $\quad 3x =$ number of leaps of the hound, 300

then $\quad 4x =$ number of leaps of the hare, 400

Let $\quad 3a =$ length of a hound leap in feet

then $\quad 2a =$ length of a hare leap in feet

$$100a + 8ax = \text{distance the hare goes in feet}$$

$$9ax = \text{distance the hound goes in feet}$$

$$\therefore\ 100a + 8ax = 9ax$$

$$x = 100$$

Explanation. Let $3x$ equal the number of leaps of the hound, then $4x$ must equal the number of leaps of the hare, because the hare makes $\frac{4}{3}$ as many leaps as the hound. Let $3a$ equal the length of a hound's leap, then $2a$ must equal the length of a hare's leap, because the length of 1 leap of the hare is $\frac{2}{3}$ times the length of 1 leap of the hound. $100a + 8ax$ equals the distance the hare goes (in 50 leaps, $100a$ and in $4x$ leaps, $8ax$); $9ax$ equals the distance the hound goes because he makes $3x$ leaps of $3a$ feet each; therefore, $100a + 8ax = 9ax$; whence, $x = 100$; $3x = 300$, the number of hound leaps and $4x = 400$, the number of hare leaps.

Proof. One hound leap $= \frac{3}{2}$ hare leaps; while the hound makes 1 leap the hare makes $\frac{4}{3}$ hare leaps; the hound gains in 1 leap $\frac{3}{2} - \frac{4}{3}$, or $\frac{1}{6}$ hare leap; the hound must leap as many times to gain 50 hare leaps as $\frac{1}{6}$ is contained times in 50, or 300 times.

Note. Sometimes the best proof of a problem that has been solved directly is its indirect solution and *vice versa.*

18. A fox is pursued by a greyhound and has a start of 121 of her own leaps; the fox makes 6 leaps while the hound makes 5; but the hound in 4 leaps goes as far as the fox in 7. How many leaps does each make before the hound catches the fox?

19. Is it possible in Problem 18 to determine the length of one leap of the hound? Of one leap of the fox?

SOLUTION OF PROBLEMS

20. A coach sets out and travels at the rate of 47 miles in 5 hours; 30 minutes later a second coach starts out from the same place and travels in the same direction, at the rate of 47 miles in $4\frac{1}{2}$ hours. In how many hours will the second coach overtake the first?

21. At what time are the hands of a watch together between 3 and 4? At what time between the same hours are they at right angles?

22. A wheelman set out from B at the rate of r miles an hour; a hours later another started in pursuit at the faster rate of p miles an hour. How far from B will the second overtake the first? What will be the distance, if $r = 10$, $p = 12$, and $a = 8$?

23. At what time is the hour hand of a clock as many minute spaces after 5 as the minute hand is spaces before 10?

24. Two men, A and B, 57 miles apart, travel toward each other, A at the rate of 6 miles an hour and B at the rate of 5 miles an hour, B starting 20 minutes later than A. How far will each have traveled when they meet?

25. A can row 4 miles an hour and B 3 miles an hour in still water; A is 14 miles farther upstream than B, and they row toward each other till they meet, 4 miles above B's starting place. Find the rate of the current.

26. Find the time between 4 and 5 o'clock when the minute hand of a watch is 18 minute spaces in advance of the hour hand.

27. In a handicap race the first horse has a start of 30 of his own leaps; he takes 6 leaps to the second horse's 5, and 7 of the leaps of the second horse are equivalent to 9 leaps of the first horse. How many leaps will the first horse take before he is overtaken by the second horse?

SIMULTANEOUS EQUATIONS
FIRST DEGREE

DEFINITIONS

Simultaneous equations are equations which are satisfied by the same values of the unknown quantities. Independent equations are equations which cannot be reduced to the same form.

$x + y = 8$ (1)	$x + y = 7$ (3)	$3x + 3y = 9$ (5)
$x - y = 4$ (2)	$x + y = 3$ (4)	$2x + 2y = 6$ (6)

Equations (1) and (2) are simultaneous equations, because they are satisfied by the same values of the unknown quantities; $x = 6$ and $y = 2$ in both equations.

Equations (3) and (4) are not simultaneous, because they are not satisfied by the same values of x and y; if $x = 5$ and $y = 2$ in (3), these values cannot hold in (4). No other values for x and y can be substituted in (3) which will hold in (4).

Equations (5) and (6) are simultaneous, but not independent because each can be reduced to $x + y = 3$.

Simultaneous equations admit of definite solutions when there are as many independent equations as there are unknown quantities.

$x + y = 8$ is one equation with two unknown quantities. No definite solution is possible. If $x = 1$, $y = 7$; if $x = 2$, $y = 6$; if $x = 10$, $y = -2$, and so on; there are an infinite number of values for x and a corresponding number of values for y.

Equations (1) and (2) are two independent equations with two unknown quantities. There is only one value for x and only one value for y. A definite solution is therefore possible.

SOLUTION—BY ADDITION OR SUBTRACTION

If there are n equations with n unknown quantities, the first step is to get $n-1$ equations with $n-1$ unknown quantities; the second step is to get $n-2$ equations with $n-2$ unknown quantities; and so on until there is one equation with one unknown quantity.

Each step may be taken in one of three ways: by addition or subtraction, by substitution, or by comparison.

Every equation must be used. The solution should be planned before anything is written.

Thus, if there are 5 equations with 5 unknown quantities, the first step is to get 4 equations with 4 unknown quantities; the second step is to get 3 equations with 3 unknown quantities; and so on until there is 1 equation with one unknown quantity.

1. Find the values of x, y, and z by addition or subtraction:

$$2x-3y+z=-1 \qquad (1)$$
$$3x+y-4z=-7 \qquad (2)$$
$$5x-y+3z=12 \qquad (3)$$

Plan. Since we have 3 equations with 3 unknown quantities, the first step is to get 2 equations with 2 unknown quantities, and the next step is to get 1 equation with 1 unknown quantity. We will take each step by addition and subtraction, eliminating x from (1) and (2) and from (1) and (3), and z from the resulting equations.

First Step. We see that x will be eliminated from (1) and (2) if (1) is multiplied by 3, if (2) is multiplied by 2, and if one result is subtracted from the other:

$$\begin{array}{r} 6x-9y+3z=-3 \\ 6x+2y-8z=-14 \\ \hline 11y-11z=-11 \\ y-z=-1 \end{array} \qquad (4)$$

We subtract the upper from the lower in this instance, because it is more convenient to have the first term positive. Dividing the remainder by 11, we obtain (4), one of the two equations with two unknown quantities.

We see that x will be eliminated from (1) and (3) if (1) is multiplied by 5, if (3) is multiplied by 2, and if one result is subtracted from the other:

$$\begin{array}{r} 10x - 15y + 5z = -\ 5 \\ 10x - \ \ 2y + 6z = \ \ 24 \\ \hline 13y + \ \ z = \ \ 29 \end{array} \qquad (5)$$

The first step is now completed; we have two equations with two unknown quantities:

$$y - z = -1 \qquad (4)$$

$$13y + z = 29 \qquad (5)$$

Second Step. Eliminating z from (4) and (5) by addition, we complete the second step; we have one equation with one unknown quantity.

$$14y = 28;\ \ y = 2$$

Completion. This value of y may be substituted in (4) or in (5); we will choose (4):

$$2 - z = -1;\ \ z = 3$$

These values of y and z may be substituted in (1), (2), or (3); we will choose (1):

$$2x - 6 + 3 = -1;\ \ x = 1$$

Proof. Substituting the values of x, y, and z in each equation,

$$2 - 6 + 3 = -1 \qquad (1)$$

$$3 + 2 - 12 = -7 \qquad (2)$$

$$5 - 2 + 9 = 12 \qquad (3)$$

Note. The work of elimination may be performed on a separate slip of paper and only the results preserved. Thus:

$$2x - 3y + z = -1 \qquad (1)$$

$$3x + y - 4z = -7 \qquad (2)$$

$$5x - y + 3z = 12 \qquad (3)$$

$$y - z = -1 \qquad (4)$$

$$13y + z = 29 \qquad (5)$$

$$y = 2,\ \ z = 3,\ \ x = 1$$

EXAMPLES FOR THE THREE METHODS

The pupil should solve the following examples by addition or subtraction. After mastering solutions by comparison, he should solve them all again by comparison. After mastering solutions by substitution, he should solve them all again by substitution.

2. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

3. $\begin{cases} 3x + 4z = 18 \\ x + 2z = 8 \end{cases}$

4. $\begin{cases} 2x - y = 9 \\ 3x - 7y = 19 \end{cases}$

5. $\begin{cases} \frac{y}{4} + \frac{z}{3} = 9 \\ \frac{y}{3} - \frac{z}{6} = 1 \end{cases}$

6. $\begin{cases} x + 3y + 4z = 14 \\ x + 2y + z = 7 \\ x + y + 2z = 4 \end{cases}$

7. $\begin{cases} 2x + y = 4 \\ 7x + 8y = -13 \end{cases}$

8. $\begin{cases} 8x - y = -6 \\ x + 8y = -17 \end{cases}$

9. $\begin{cases} 5x - 3y = 4 \\ 7x - 12y = -10 \end{cases}$

10. $\begin{cases} \frac{x}{2} + \frac{2y}{3} = 16 \\ \frac{x}{4} - \frac{5y}{6} = -6 \end{cases}$

11. $\begin{cases} x + 2y + 2z = 11 \\ 2x + y + z = 7 \\ 3x + 4y + z = 14 \end{cases}$

12. If there are 4 equations with 4 unknown quantities, what are the steps?

13. If the equations are numbered (1), (2), (3), and (4), what equations may be combined to get (5), or the first equation of the second set?

14. To get (6), or the second equation of the second set? To get (7)?

NOTE. The first step in Exs. 5 and 10 is to clear of fractions.

SOLUTION—BY COMPARISON

To eliminate by comparison is to find the value of the same unknown quantity, in terms of the other unknown quantities, in each of two equations and to place these values equal to each other.

15. Find the values of x, y, and z by comparison:

$$2x - 3y + z = -1 \quad (1)$$
$$3x + y - 4z = -7 \quad (2)$$
$$5x - y + 3z = 12 \quad (3)$$

Plan. Since we have 3 equations with 3 unknown quantities, the first step is to get 2 equations with 2 unknown quantities, and the next step is to get one equation with one unknown quantity. We will take each step by comparison, eliminating x from (1) and (2) and from (1) and (3), and z from the resulting equations.

First Step. We see that x will be eliminated from (1) and (2) if its value in the one is placed equal to its value in the other.

$$x = \frac{3y - z - 1}{2} \quad (1) \qquad x = \frac{-y + 4z - 7}{3} \quad (2)$$

$$\therefore \quad \frac{3y - z - 1}{2} = \frac{-y + 4z - 7}{3}$$

Simplifying, $$y - z = -1 \quad (4)$$

We will eliminate x from (1) and (3) in a similar manner:

$$x = \frac{3y - z - 1}{2} \quad (1) \qquad x = \frac{y - 3z + 12}{5} \quad (3)$$

$$\therefore \quad \frac{3y - z - 1}{2} = \frac{y - 3z + 12}{5}$$

Simplifying, $$13y + z = 29 \quad (5)$$

Second Step. We will eliminate z from (4) and (5) in a similar manner:

$$z = y + 1 \quad (4) \qquad z = 29 - 13y \quad (5)$$

$$y + 1 = 29 - 13y; \quad 14y = 28; \quad y = 2$$

The pupil should solve Exs. 2 to 11 by comparison.

SOLUTION — BY SUBSTITUTION

To eliminate by substitution is to find the value of an unknown quantity, in terms of the other unknown quantities, in one of two equations and to substitute this value in the other equation.

16. Find the values of x, y, and z by substitution:

$$2x - 3y + z = -1 \qquad (1)$$
$$3x + y - 4z = -7 \qquad (2)$$
$$5x - y + 3z = 12 \qquad (3)$$

PLAN. Since we have 3 equations with 3 unknown quantities, the first step is to get 2 equations with 2 unknown quantities, and the next step is to get one equation with one unknown quantity. We will take each step by substitution, eliminating x from (1) and (2) and from (1) and (3), and z from the resulting equations.

FIRST STEP. We see that x will be eliminated from (1) and (2) if its value in the one is substituted in the other. We will find the value of x in (1) and substitute in (2).

$$x = \frac{3y - z - 1}{2} \qquad (1)$$

Substituting, $$\frac{3(3y - z - 1)}{2} + y - 4z = -7$$

Simplifying, $$y - z = -1 \qquad (4)$$

We will eliminate x from (1) and (3) by substituting in (3) the value of x found in (1).

Substituting, $$\frac{5(3y - z - 1)}{2} - y + 3z = 12$$

Simplifying, $$13y + z = 29 \qquad (5)$$

SECOND STEP. We will eliminate z from (4) and (5) by finding the value of z in (4) and substituting in (5):

$$z = y + 1 \qquad (4)$$

$$13y + y + 1 = 29;\ 14y = 28;\ y = 2$$

The pupil should solve Exs. 2 to 11 by substitution.

SOLUTION — DISCUSSION

It is interesting to note that an equation of the form $0 = 0$ results if any one of the equations is neglected in the process of elimination. All of the equations must be used.

17. Eliminate x from (1) and (2), (1) and (3), and (2) and (3). Number the resulting equations (5), (6), and (7):

$$x + y + z + w = 4 \qquad (1)$$
$$x + 2y + 3z + 4w = 10 \qquad (2)$$
$$x - y + 2z - 3w = -1 \qquad (3)$$
$$x + 2y - z + 3w = 5 \qquad (4)$$

NOTE. A mistake has been made because (4) has not been used. (1) and (4), (2) and (4), or (3) and (4) should have been chosen instead of (2) and (3).

18. From the set found in Ex. 17, eliminate y from (5) and (6) and from (5) and (7). Number the resulting equations (8) and (9).

19. Eliminate z from the set found in Ex. 18. What is the trouble?

20. Go back to Ex. 17 and correct the error, *i.e.* get (7) by combining (1) and (4); then find the values of x, y, z, and w.

Through mistakes, values are sometimes found which satisfy one or more but not all of the equations. In the proof, every equation must be satisfied.

21. If $x = -\frac{5}{11}$, $y = \frac{33}{11}$, $z = \frac{15}{11}$, $w = \frac{1}{11}$, is equation (1) satisfied?

22. Is equation (2) satisfied by the values of the unknown quantities given in Ex. 21? Is equation (3) satisfied?

23. Is equation (4) satisfied? What conclusion do you reach from a study of the last three examples?

24. Which of the three methods of elimination is generally to be preferred?

SOLUTION — MISCELLANEOUS

Find the values of the unknown quantities:

25. $\begin{cases} 2x + 3y = 12 \\ 3x + 2y = 13 \end{cases}$

26. $\begin{cases} 4x + 3y = 22 \\ 3x - 3y = 6 \end{cases}$

27. $\begin{cases} 6x - 5y = 7 \\ 15x - 7y = 23 \end{cases}$

28. $\begin{cases} 69x - 17y = 1 \\ 13x + 14y = 69 \end{cases}$

29. $\begin{cases} ax + by = a^2 + b^2 \\ bx + ay = 2ab \end{cases}$

30. $\begin{cases} mx - ny = m \\ ax - by = a \end{cases}$

31. $\begin{cases} ax - by = o \\ mx + ny = p \end{cases}$

32. $\begin{cases} ax + by = a^2 \\ bx + ay = b^2 \end{cases}$

33. $\begin{cases} (a+b)x - (a-b)y = 4ab \\ (a-b)x - (a+b)y = o \end{cases}$

34. $\begin{cases} (a-b)x + (a+b)y = 2a^2 - 2b^2 \\ (a+b)x - (a-b)y = 4ab \end{cases}$

35. $\begin{cases} 4x - 3y - 2z = 4 \\ 3x - 4y + 6z = 7 \\ 2x + 3y - 4z = 8 \end{cases}$

36. $\begin{cases} x + y + z = 3 \\ x + 2y + 3z = 4 \\ 6x + 4y + 3z = 16 \end{cases}$

37. $\begin{cases} 5z + 4x - 7y = 2 \\ 6x + 4y - 6z = 4 \\ 6y - z + 7x = 12 \end{cases}$

38. $\begin{cases} 4y + 5z + 2x = 22 \\ -3x + 5y + 7z = 18 \\ -3y + 8x + 5z = 20 \end{cases}$

All of the unknown quantities may be found by addition and subtraction.

Ex. 30. TO ELIMINATE y

$$
\begin{array}{lrl}
(1)\times b, & bmx - bny &= bm \\
(2)\times n, & anx - bny &= an \\
\hline
& (bm - an)x &= bm - an \\
& x &= 1
\end{array}
$$

TO ELIMINATE x

$$
\begin{array}{lrl}
(1)\times a, & amx - any &= am \\
(2)\times m, & amx - bmy &= am \\
\hline
& (an - bm)y &= 0 \\
& y &= 0
\end{array}
$$

NOTE. The pupil should subtract as on p. 25. It is not in good form to indicate the subtraction, $bmx - anx = bm - an$, and then to factor, $(bm - an)x = bm - an$.

Ex. 34. TO ELIMINATE y

$$
\begin{array}{ll}
(1)\times(a-b), & (a^2 - 2ab + b^2)x + (a^2 - b^2)y = 2a^3 - 2a^2b - 2ab^2 + 2b^3 \\
(2)\times(a+b), & (a^2 + 2ab + b^2)x - (a^2 - b^2)y = \qquad 4a^2b + 4ab^2 \\
\hline
& (2a^2 + 2b^2)x \qquad = 2a^3 + 2a^2b + 2ab^2 + 2b^3 \\
& x = a + b
\end{array}
$$

TO ELIMINATE x

$$
\begin{array}{ll}
(1)\times(a+b), & (a^2 - b^2)x + (a^2 + 2ab + b^2)y = 2a^3 + 2a^2b - 2ab^2 - 2b^3 \\
(2)\times(a-b), & (a^2 - b^2)x - (a^2 - 2ab + b^2)y = \qquad 4a^2b - 4ab^2 \\
\hline
& (2a^2 + 2b^2)y = 2a^3 - 2a^2b + 2ab^2 - 2b^3 \\
& y = a - b
\end{array}
$$

Sometimes the addition or subtraction of the equations member by member will give simpler equations.

Ex. 34.

$$2ax + 2by = 2a^2 + 4ab - 2b^2 \qquad (1)$$

$$2bx - 2ay = -2a^2 + 4ab + 2b^2 \qquad (2)$$

Adding, we get (1); subtracting the upper from the lower, we get (2).

MISCELLANEOUS

Find the values of the unknown quantities:

39. $\begin{cases} x+y=13 \\ x+z=14 \\ y+z=15 \end{cases}$

40. $\begin{cases} \frac{5}{x}+\frac{3}{y}=30 \\ \frac{9}{x}-\frac{5}{y}=2 \end{cases}$

41. $\begin{cases} \frac{1}{x}+\frac{2}{y}=2 \\ \frac{3}{y}-\frac{4}{z}=\frac{1}{6} \\ \frac{3}{z}-\frac{4}{x}=-3 \end{cases}$

42. $\begin{cases} \frac{2x-y}{3}=\frac{5}{3} \\ \frac{3y-2z}{4}=\frac{5}{4} \\ \frac{x+2z}{5}=\frac{8}{5} \end{cases}$

43. $\begin{cases} x-y=18 \\ x-z=33 \\ y+z=25 \end{cases}$

44. $\begin{cases} \frac{x+y}{2}-\frac{x-y}{3}=8 \\ \frac{x+y}{3}+\frac{x-y}{4}=11 \end{cases}$

45. $\begin{cases} \frac{1}{x}+\frac{2}{y}+\frac{3}{z}=6 \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3 \\ \frac{1}{x}-\frac{1}{y}+\frac{5}{z}=5 \end{cases}$

46. $\begin{cases} \frac{3}{x}-\frac{4}{3y}+\frac{1}{z}=\frac{4}{3} \\ \frac{2}{x}+\frac{3}{y}-\frac{1}{z}=\frac{7}{6} \\ \frac{3}{x}-\frac{2}{3y}+\frac{2}{z}=\frac{8}{3} \end{cases}$

Ex. 40. First Solution. Such equations are usually solved without clearing of fractions by regarding $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities. Multiplying (1) by 9 and (2) by 5, $\frac{45}{x}+\frac{27}{y}=270$, $\frac{45}{x}-\frac{25}{y}=10$; subtracting, $\frac{52}{y}=260$; $y=\frac{1}{5}$. Ex. 41, 45, and 46 should be solved in this way.

Second Solution. Clearing of fractions, $5y+3x=30xy$ (1); $9y-5x=2xy$ (2); multiplying (1) by 9, (2) by 5, and subtracting, $52x=260xy$; dividing by x, $52=260y$, $y=\frac{1}{5}$.

SOLUTION OF PROBLEMS

In the direct solution of problems, as many unknown quantities may be used as there are conditions, but the use of one unknown quantity, as in the problems thus far given, is often to be preferred.

1. The sum of two numbers is 8; the larger is 3 times the smaller. Find the numbers.

ONE UNKNOWN

Let $x =$ the smaller, 2

then $3x =$ the larger, 6

$4x =$ the sum

$4x = 8$

$x = 2$

TWO UNKNOWN

Let $x =$ the smaller, 2

and $y =$ the larger, 6

$$y = 3x \qquad (1)$$

$$x + y = 8 \qquad (2)$$

$$x = 2,\ y = 6$$

2. Two times the larger of two numbers minus 3 times the smaller is 20; 3 times the larger plus 2 times the smaller is 43. Find the numbers.

ONE UNKNOWN

Let $x =$ the larger, 13

then $\dfrac{2x-20}{3} =$ the smaller, 2

$$3x + \frac{4x-40}{3} = 43$$

$$x = 13$$

TWO UNKNOWN

Let $x =$ the larger, 13

and $y =$ the smaller, 2

$$2x - 3y = 20 \qquad (1)$$

$$3x + 2y = 43 \qquad (2)$$

$$x = 13$$

$$y = 2$$

3. Which solution do you prefer for the first problem? Which, for the second?

4. (*a*) Write a problem which may be solved by the equations of Ex. 2, p. 110:

(*b*) Solve it by the use of one unknown quantity; by the use of two.

(*c*) The sum of the equations involving the two unknown quantities is $2x = 6$; their difference, $2y = 4$. What principles may be deduced from the last two equations?

SOLUTION OF PROBLEMS

5. (*a*) A bought 5 calves and 6 sheep for \$89, and at the same prices 7 calves and 8 sheep for \$121. What was the price of each per head?

Let x = cost in \$ of 1 calf, 7

and y = cost in \$ of 1 sheep, 9

$$5x + 6y = 89 \quad (1)$$
$$7x + 8y = 121 \quad (2)$$
$$x = 7, \ y = 9$$

(*b*) Solve by the use of one unknown quantity.

6. Five bushels of corn, 6 bushels of oats, and 8 bushels of rye together are worth \$10.30; 3 bushels of corn, 5 bushels of oats, and 8 bushels of rye at the same prices, \$8.75; 1 bushel of oats mixed with 1 bushel of rye at the same prices is worth as much as $1\frac{7}{8}$ bushels of corn. What is the value of each per bushel?

7. A boy bought 2 apples and 5 pears for 12¢; at the same prices, 3 pears and 4 peaches for 18¢; at the same prices, 4 pears and 5 oranges for 28¢; at the same prices, 6 peaches and 5 oranges for 38¢. Required the cost of each kind of fruit.

8. The cost of a lb. of tea with b lb. of sugar is m cents; at the same prices the cost of c lb. of tea with d lb. of sugar is n cents. What is the cost of each per pound?

9. A, B, and C in partnership gain \$960; A owns $\frac{1}{4}$ of the stock, plus \$1000, B's gain is \$240, and C's \$420. Required each one's share of stock. Let x = entire stock in dollars.

10. A man bought a horse, carriage, and harness for \$500; the horse cost \$5 more than the carriage and harness, and the carriage cost $\frac{2}{3}$ as much as the horse and harness. Required the cost of each.

11. For \$2500 I bought 8 horses, a number of cows at \$40, and 100 sheep; the number of cows was equal numerically to 4 times the price of a sheep in dollars; a sheep and a horse together cost \$5 less than $\frac{1}{5}$ the cost of all the cows. Find the cost of a horse.

12. Divide the number 89 into two parts such that $\frac{1}{8}$ of the greater part will exceed $\frac{1}{2}$ of the less by 1.

13. Two purses contain together \$300; if \$30 is taken from the first and put into the second, there will be the same amount in each. How much money is there in each?

14. The sum of the ages of a father and son is 80 years; if the age of the son is doubled, it will exceed the age of the father by 10 years. What is the age of each?

15. A banker has two kinds of money; it takes 10 pieces of one kind or 4 pieces of the other to make a dollar. If 7 pieces are given for a dollar, how many of each will be used?

16. A banker has two kinds of money; it takes a pieces of one kind or b pieces of the other to make a dollar. If c pieces are given for a dollar, how many of each will be used?

17. If A were to receive \$10 from B, he would then have twice as much as B would have left; but if B were to receive \$10 from A, B would have three times as much as A would have left. How much has each?

18. The sum of $\frac{1}{2}$ of one number and $\frac{2}{3}$ of another is 38; and if 3 be added to the first, the sum will be equal to $\frac{5}{8}$ of the difference between the second and 8. Find the numbers.

19. A and B can do a certain work in 24 days, B and C in 40 days, A and C in 30 days. In how many days can each do the work?

20. A and B can together perform a certain work in 30 days; at the end of 18 days B is called off, and A finishes it alone in 20 more days. Find the time in which each can perform the work alone.

SOLUTION OF PROBLEMS

In problems involving the digits of a number, it is well to represent units' digit by x, tens' digit by y, hundreds' digit by z, $\cdots$.

21. If x represents units' digit; y, tens' digit; and z, hundreds' digit; what represents the number? What represents the number in which the same digits occur in reverse order?

22. The sum of the two digits of a number is 11; if 9 is added to the number, the digits will occur in reverse order. Find the number.

23. The sum of the three digits of a number is 10; if 297 is added to the number, the digits will occur in reverse order; units' digit is equal to the sum of the other two. Find the number.

24. The sum of the two digits of a certain number is 6 times their difference, and the number itself exceeds 6 times their sum by 3. Find the number.

25. A certain number is expressed by three digits whose sum is 10; the sum of units' and hundreds' digits is 4 times tens' digit; and if 198 be subtracted from the number, the digits will be reversed. What is the number?

26. The sum of the four digits of a number is 17; either of the two middle digits is units' digit minus thousands' digit; if hundreds' digit is transposed with units' digit, the number is increased by 396. Find the number.

27. Make up a problem which may be solved by the equations of Ex. 44, p. 116.

28. My agent sold my flour at 4% commission. Increasing the proceeds by \$168, I bought wheat, paying 2% commission. Wheat declining 3%, my loss, including commissions, was \$30. What was the selling price of the flour?

29. At simple interest, the amount of a certain principal for m years is a dollars and for n years b dollars. Find the principal and the rate.

INDETERMINATE EQUATIONS

When the number of equations is less than the number of unknown quantities, the equations are indeterminate (p. 107), but conditions may be imposed which limit the number of values of the unknown quantities. The usual condition is that the unknown quantities shall be positive integers.

Indeterminate equations are solved by trial. The obvious method is to combine the equations so as to obtain one equation with as few unknown quantities as possible, and to substitute the positive integers 1, 2, 3, $\cdots$; but the devices stated in the next two propositions are often employed before the trial is made.

PROPOSITION XLVIII. THEOREM

When x and y are positive integers, if the value of x is found in terms of y and reduced to a mixed expression, the fractional part must equal a positive integer, a negative integer, or zero.

Let
$$x = a + by + \frac{c + ny}{s} \qquad (1)$$

To Prove that $\frac{c + ny}{s}$ must equal a positive integer, a negative integer, or zero when x and y are positive integers.

Transposing (1), $\quad x - a - by = \frac{c + ny}{s}$

If $x - a > by$, $\frac{c + ny}{s} =$ a positive integer; if $x - a < by$, $\frac{c + ny}{s} =$ a negative integer; if $x - a = by$, $\frac{c + ny}{s} = 0$.

Hence, the principle, Q.E.D.

NOTE. In complicated problems, the devices of this and the following proposition simplify the labor, but the method of direct trial is often the shorter.

INDETERMINATE EQUATIONS

Proposition XLIX. Theorem

When x and y are positive integers, if the value of x is found in terms of y and reduced to a mixed expression, and if the fractional part is multiplied by a positive integer or by a negative integer, the result is equal to a positive integer, a negative integer, or zero.

By the previous proposition, the fractional part is equal to a positive integer, a negative integer, or zero. If the multiplier is a positive integer, the product will be an integer of the same sign as before, or zero; if the multiplier is a negative integer, the product will be an integer with a different sign than before, or zero.

Hence, the principle, Q.E.D.

30. Solve $3x + 5y = 34$ in positive integers without the use of the devices stated in the foregoing propositions.

$x = \frac{34 - 5y}{3}$; if $y = 1$, $x =$ a fraction; if $y = 2$, $x = 8$; if $y = 3$ or 4, $x =$ a fraction; if $y = 5$, $x = 3$; if $y = 6$, $x =$ a fraction; if $y > 6$, x is negative. Hence, the only positive integral values are: $x = 8, 3$; $y = 2, 5$.

31. Solve $3x + 5y = 34$ in positive integers and make use of the devices stated in the foregoing propositions.

$$x = \frac{34 - 5y}{3} = 11 - y + \frac{1 - 2y}{3}.$$

By XLVIII, $\frac{1 - 2y}{3}$ is equal to a positive integer, a negative integer, or zero. If we multiply by something that will make the coefficient of y greater by 1 than some multiple of the denominator, the result will be equal to a positive integer, a negative integer, or zero, and can be reduced to a mixed number whose fractional part will have y with a coefficient $+1$ or -1. Multiplying by 2, we get $\frac{2 - 4y}{3}$, or $-y + \frac{2 - y}{3}$, of which, by XLIX, $\frac{2 - y}{3}$ must be a positive integer, a negative integer, or zero.

Let $\frac{2-y}{3}=m$, a positive integer, a negative integer, or zero. Simplifying, $y=2-3m$; substituting this value of y in the original equation, $x=8+5m$. The advantage of having the coefficient of y, $+1$, or -1, is now apparent; if it had been any other integer, the value of y would have been in fractional form.

$$y=2-3m \qquad (1)$$

$$x=8+5m \qquad (2)$$

Thus far, no trial has been made. We have found the values of x and of y in terms of the same positive integer, the same negative integer, or zero. We will now substitute for m in both equations all positive integers, all negative integers, and zero.

Positive Integers. If $m=1$ or any integer greater than 1, y is negative. Hence, m cannot be a positive integer.

Negative Integers. If $m=-1$, y becomes 5, and x becomes 3; if $m=-2$, or any other negative integer, x becomes negative. Hence, m may be -1, but no other negative integer.

Zero. If $m=0$, $y=2$, and $x=8$. Hence, m may be 0.

The only possible values of m are -1 and 0; of y, 5 or 2; of x, 3 or 8.

By aid of the devices, solve in positive integers:

32. $5x+12y=263$

33. $8x-3y=28$

34. $3x-14y=11$

35. $2x+5y=40$

36. $\begin{cases}2x+3y-5z=-8\\5x-y+4z=21\end{cases}$

37. $\begin{cases}2x+5y+7z=80\\3x+2y+6z=90\end{cases}$

Ex. 36. Eliminate one of the unknown quantities, and find the values of x, y, and z in terms of m. Thus,

Eliminating y, $\qquad 7z+17x=55$, or $z=\frac{55-17x}{7}$

Reducing, $\qquad z=7-2x+\frac{6-3x}{7}$

Simplifying, $\qquad \frac{6-3x}{7}\times 5=\frac{30-15x}{7}$, or $4-2x+\frac{2-x}{7}$

Let $\qquad \frac{2-x}{7}=m$

then, $\qquad x=2-7m$; $z=3+17m$; $y=1+33m$

whence, $\qquad x=2$, $z=3$, $y=1$

INDETERMINATE EQUATIONS—PROBLEMS

Alligation treats of mixing or compounding two or more ingredients of different values or quantities. Solutions are complicated by the indirect process, but simple by the direct.

1. A grocer wishes to make a mixture of 50 lb. of tea worth 80¢ a pound. He wishes to use 20 lb. worth 75¢ a pound and an integral number of pounds of teas worth 40¢, 60¢, and 90¢ respectively. How much of each kind may he use? Prove.

Let x, y, and z equal the pounds of each

$$20 + x + y + z = 50 \tag{1}$$

$$20 \times 75 + 40x + 60y + 90z = 80(20 + x + y + z) \tag{2}$$

$$2y + 5z = 130 \tag{3}$$

$$x = 1,\ y = 5,\ z = 24$$

2. A miller wishes to grind together 48 cwt. of corn worth 90¢ a hundredweight, oats worth 60¢ a hundredweight, and barley worth \$1.68 a hundredweight to make a mixture worth \$1.20 a hundredweight. How many integral hundredweight of each may he use?

3. How much wheat at \$1.00 and at \$.90 per bushel must be mixed with 4 bushels at \$1.10 a bushel, and 5 bushels at \$.80, so as to form a mixture of 24 bushels at \$.95 a bushel?

4. A grain dealer has clover seed at \$7 a bushel, and blue-grass seed at \$8.50. How much of each must he sell in order to realize an average price of \$8.00 a bushel?

5. How many pounds of tea at 15 cents, 17 cents, and 22 cents a pound, must be mixed with 5 pounds of tea at 20 cents a pound to make a mixture of 75 pounds at 18 cents a pound?

6. How much water, and how much wine at 90¢ a gallon, must be mixed to produce 30 gal. at 60¢ a gallon?

7. What is the least number which, when divided by 3 and 5, leaves remainders of 2 and 3 respectively?

Let $x =$ the number, 8

and $y =$ first quotient, 2

and $z =$ second quotient, 1

$$\frac{x-2}{3} = y \qquad (1)$$

$$\frac{x-3}{5} = z \qquad (2)$$

Clearing (1), $x - 3y = 2$

Clearing (2), $x - 5z = 3$

$$3y - 5z = 1$$

$$y = \frac{5z+1}{3}$$

Least values: $x = 8;\ y = 2;\ z = 1$

8. Find three positive integral numbers whose sum is 18.

9. Divide 55 into two parts such that one shall be divisible by 2, and the other by 3.

10. What is the least number which, when divided by 7, 8, and 9 respectively, leaves a remainder of 1?

11. The sum of the numerator and the denominator of a fraction is 83; the numerator is divisible by 5 and the denominator by 7. Find the fraction.

12. A number is expressed by 3 digits whose sum is 20; if 16 is subtracted from the number and the remainder divided by 2, the digits will be reversed. What is the number?

13. Divide 15 into three parts such that, if the first be multiplied by 2, the second by 3, and the third by 2, the sum of the products will be 32.

14. There are three positive integral numbers whose sum is 33; three times the first plus twice the second plus four times the third is 127. What are the numbers?

INDETERMINATE EQUATIONS — PROBLEMS

15. In how many ways can \$100 be paid with \$10 and \$5 bills, by the use of both kinds at each payment?

16. When the numerator of a certain fraction is doubled and its denominator is increased by 7, its value becomes $\frac{2}{3}$. Find the fraction.

17. A farmer purchased a certain number of pigs, sheep, and calves for \$160; the pigs cost \$3 each, the sheep \$4 each, and the calves \$7 each; and the number of calves was equal to the number of pigs and sheep together. How many of each did he buy?

18. How many calves at \$6 a head and sheep at \$8 a head can be bought for \$180?

19. A farmer spends \$752 in buying horses and cows; each horse costs \$37, and each cow \$23. How many of each does he buy?

20. How many calves at \$7, sheep at \$3, and lambs at \$1 per head can be bought for \$200, if the whole number of animals is 100?

21. A number of lengths are 3 ft., 5 ft., and 8 ft. How may 48 of them be taken so as to measure 175 ft. all together?

22. A in 2 days, B in 3 days, and C in 4 days, together earn \$47; A in 3 days, B in 4 days, and C in 6 days, earn \$68. What are the daily wages of each, supposing them to be an integral number of dollars?

23. There are 3 positive integral numbers whose sum is 40; 2 times the first plus 5 times the second plus 4 times the third is 128. What are the numbers?

24. There is a bookshelf which carries 20 books. When books are composed of sets of 5 volumes each, of 4 each, and of 3 each, find how they must be distributed so that no set is divided, and so that at least 1 set of each is upon the shelf.

INVOLUTION, EVOLUTION, LOGARITHMS

DEVELOPMENT

When a quantity is taken several times as an addend, the result the same number of times, the last result the same number of times, and so on, the fact that the whole is equal to the sum of all its parts may be illustrated:

$$2a + 2a + 2a = 3 \cdot 2a$$

$$3 \cdot 2a + 3 \cdot 2a + 3 \cdot 2a = 3 \cdot 3 \cdot 2a = 3^2 \cdot 2a, \text{ or } 9 \times 2a = 3^2 \times 2a$$

$$3^2 \cdot 2a + 3^2 \cdot 2a + 3^2 \cdot 2a = 3 \cdot 3^2 \cdot 2a = 3^3 \cdot 2a, \text{ or } 27 \times 2a = 3^3 \times 2a$$

$$3^3 \cdot 2a + 3^3 \cdot 2a + 3^3 \cdot 2a = 3 \cdot 3^3 \cdot 2a = 3^4 \cdot 2a, \text{ or } 81 \times 2a = 3^4 \times 2a$$

If both members of the equation $81 \times 2a = 3^4 \times 2a$ are divided by $2a$,

$$81 = 3^4$$

By the omission of each term in succession, three problems arise:

1. What $= 3^4$?
2. $81 =$ what ^{4}?, or what $= \sqrt[4]{81}$?
3. $81 = 3^{\text{what}}$?, or what $= \log_3 81$?

Problem 1 gives rise to involution. It means, what used once as a multiplier will produce the same result as 3 used 4 times as

DEVELOPMENT

a multiplier? Involution is the process of finding a multiplier that will produce the same result as a given number used a given number of times as a multiplier. The common definition, "Involution is the process of raising a number to any power," must be understood in this sense. The multiplier required is the power; the number used as a multiplier, the base; the number denoting how many times the base is used as a multiplier, the exponent. All of the terms denote *times*. Involution is expressed by writing the exponent above and a little to the right of the base; the exponent of the exponent may also be expressed. Thus:

$$\text{The 4th power of } 3 = 3^4$$
$$\text{The 4th power of } 3 = 3^{(4^{+1})}$$

Problem 2 gives rise to evolution. It means, what number must be used 4 times as a multiplier to produce the same result as 81 used once as a multiplier? Evolution is the process of finding a number that, when used a given number of times as a multiplier, will produce the same result as a given number used once as a multiplier. The common definition, "Evolution is the process of depressing a number to a given root," must be accepted in this sense. The required multiplier is the root; the number denoting how many times the root is to be used as a multiplier, the index; the given multiplier, the base. All of the terms denote *times*. Evolution is expressed by placing the base under and the index within the symbol, $\sqrt{\ }$, by the denominator of a fractional exponent, or by an exponent whose exponent is -1. Thus:

$$\text{The 4th root of } 81 = \sqrt[4]{81}$$
$$\text{The 4th root of } 81 = 81^{\frac{1}{4}}$$
$$\text{The 4th root of } 81 = 81^{(4^{-1})}$$

Problem 3 gives rise to logarithms. It means, how many times must 3 be used as a multiplier to produce the same result as 81

used once as a multiplier? A logarithm denotes the number of times a given number must be used as a multiplier to produce the same result as a given number used once as a multiplier. The common definition, "The logarithm of a number is the exponent of the power to which a base must be raised to produce a given number," must be accepted in this sense. The number that is to be used as a multiplier is the base. All of the terms denote *times*. The logarithm of a number is expressed by writing *log* before the number and by writing the base a little below and to the right of the number. Thus:

The logarithm of 81 in the system whose base is $3 = \log_3 81$

1. What is the result when a is taken 2 times as an addend, the result 2 times as an addend, the last result 2 times as an addend, and so on until the operation has been performed 5 times?

Ans. $a + a = 2a$; $2a + 2a = 2 \cdot 2a = 2^2 a$; $2^2 a + 2^2 a = 2 \cdot 2^2 a = 2^3 a$; $2^3 a + 2^3 a = 2 \cdot 2^3 a = 2^4 a$; $2^4 a + 2^4 a = 2 \cdot 2^4 a = 2^5 a = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2a = 32a$.

2. What is the meaning of $2^5 a = 32a$?

Ans. In terms of addition, the expression means that a taken 2 times as an addend, the result 2 times as an addend, the last result 2 times as an addend, and so on until the operation has been performed 5 times, is equal to a taken 32 times as an addend.

In terms of multiplication, the expression means that a multiplied by 2, the result multiplied by 2, and so on until the operation has been performed 5 times, is equal to a multiplied by 32.

3. In $2^5 a = 32a$, what is the meaning of each of the integers?

Ans. 2 denotes the number of times each result is to be taken as an addend. Thus, a is taken 2 times, then $2a$ is taken 2 times, then $4a$ is taken 2 times, and so on.

5 denotes the number of times the process of taking as an addend is to be performed, or the number of times 2 is to be used as a multiplier.

32 denotes the number of times a is taken as an addend, or that 32 is used once as a multiplier.

INVOLUTION — PRINCIPLES

PROPOSITION L. THEOREM*

To raise a factor to any power, write the base, and over it the product of the exponent by the number denoting the required power.

To prove $(a^m)^n = a^{mn}$

$$(a^m)^n = a^m \cdot a^m \cdot a^m \cdots n \text{ times}$$
$$= a^{m+m+m \cdots n \text{ times}} = a^{mn}$$

(To multiply when the bases are the same, write the common base, and over it the exponent of the multiplicand plus the exponent of the multiplier.)

Hence, the principle, Q.E.D.

PROPOSITION LI. THEOREM

Every even power of a negative number is '+'; every odd power of a negative number is '−.'

$$(-a)^1 = -a$$
$$(-a)^2 = -a \times -a = +a^2$$
$$(-a)^3 = -a \times -a \times -a = -a^3$$
$$(-a)^4 = -a \times -a \times -a \times -a = +a^4$$

. . . .

Hence, the principle, Q.E.D.

Prove:

1. $(ab)^3 = a^3b^3$
2. $(a^2b^2)^3 = a^6b^6$
3. $(a^mb^n)^p = a^{mp}b^{np}$
4. $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$
5. $\left(\dfrac{a^2}{b^2}\right)^3 = \dfrac{a^6}{b^6}$
6. $\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}$

* For negative and fractional exponents, see p. 179.

7. State the principle established by Ex. 3; the principle established by Ex. 6.

Simplify:

8. $(a^4b^3)^5$

9. $(-2a^2x^3)^4$

10. $(-3m^2n)^3$

11. $[(a+b)^2]^3$

12. $[(a+b)^3]^2$

13. $[(a-b)^3]^3$

Simplify:

14. $[(a^3)^4]^5$

15. $[(a^4)^3]^5$

16. $[(-a^5)^3]^7$

17. $[(-a^7)^5]^3$

18. What law do you infer from a study of these examples and their answers?

The binomial theorem (p. 55) properly belongs under this topic. It should be reviewed.

Simplify:

19. $(a+b+c)^3$

20. $(a+b+c-d)^3$

21. $(a-b+c)^4$

22. $(2a^2-3b^2)^3$

23. $(3a^2-4b)^4$

24. $(3x^2-1)^5$

Ex. 20. $(a+b+c-d)^3=[(a+b+c)-d]^3=(a+b+c)^3-3(a+b+c)^2d+3(a+b+c)d^2-d^3$. $(a+b+c)^3=[(a+b)+c]^3=$ etc. The resulting values should be substituted.

Ex. 22. $(2a^2-3b^2)^3=(2a^2)^3-3(2a^2)^2(3b^2)+3(2a^2)(3b^2)^2-(3b^2)^3$. It remains to simplify the terms.

25. From the answer obtained by developing $(a+b+c)^3$, state the rule for cubing any algebraic expression.

To the sum of the cubes of the several terms, add three times the square of each term by each of the terms that follow it, three times each term ….

EVOLUTION — PRINCIPLES

Proposition LII. Theorem

To depress a factor to any root, write the base and over it the quotient of the exponent by the number denoting the required root.

To prove $\sqrt[n]{a^{mn}} = a^m$

By definition, $\sqrt[n]{a^{mn}}$ calls for a number that taken n times as a multiplier will produce a^{mn}.

a^m taken n times as a multiplier will produce a^{mn}.

$$\therefore\ \sqrt[n]{a^{mn}} = a^m$$

Hence, the principle, Q.E.D.

Proposition LIII. Theorem

Every even root of a positive number is '+' or '−'; every even root of a negative number is imaginary; every odd root of a negative number is '−'.

$\sqrt{+a}$, $\sqrt[4]{+a}$, $\sqrt[6]{+a} \cdots$ each calls for a number that taken an even number of times as a multiplier will produce $+a$, or calls for either a positive or a negative number.

$\sqrt{-a}$, $\sqrt[4]{-a}$, $\sqrt[6]{-a} \cdots$ each calls for a number that taken an even number of times as a multiplier will produce $-a$, or calls for an imaginary number.

$\sqrt[3]{-a}$, $\sqrt[5]{-a}$, $\sqrt[7]{-a} \cdots$ each calls for a number that taken an odd number of times as a multiplier will produce $-a$, or calls for a negative number.

Hence, the principle, Q.E.D.

Except in the solution of equations of the second degree and in special cases, it is customary to consider the even root of an expression as positive. See p. 171.

Note. The $\sqrt{-a}$ is imaginary because $+^2 = +$ and $-^2 = +$.

Prove:

1. $\sqrt[3]{a^3b^3}=\sqrt[3]{a^3}\times\sqrt[3]{b^3}$

2. $\sqrt[3]{a^6b^6}=\sqrt[3]{a^6}\times\sqrt[3]{b^6}$

3. $\sqrt[p]{a^{mp}b^{np}}=\sqrt[p]{a^{mp}}\times\sqrt[p]{b^{np}}$

4. $\sqrt[3]{\dfrac{a^3}{b^3}}=\dfrac{\sqrt[3]{a^3}}{\sqrt[3]{b^3}}$

5. $\sqrt[3]{\dfrac{a^6}{b^6}}=\dfrac{\sqrt[3]{a^6}}{\sqrt[3]{b^6}}$

6. $\sqrt[p]{\dfrac{a^{mp}}{b^{np}}}=\dfrac{\sqrt[p]{a^{mp}}}{\sqrt[p]{b^{np}}}$

7. State the principle established by Ex. 3; the principle established by Ex. 6.

Simplify:

8. $\sqrt[5]{a^{20}b^{15}}$

9. $\sqrt[4]{16\,a^8x^{12}}$

10. $\sqrt[3]{-27\,m^6n^3}$

11. $\sqrt[3]{(a+b)^6}$

12. $\sqrt{(a+b)^6}$

13. $\sqrt[3]{(a-b)^9}$

Simplify:

14. $\sqrt[5]{\sqrt[3]{a^{60}}}$

15. $\sqrt[3]{\sqrt[5]{a^{60}}}$

16. $\sqrt[7]{\sqrt[5]{-a^{105}}}$

17. $\sqrt[5]{\sqrt[7]{-a^{105}}}$

18. What law do you infer from a study of these examples and answers?

Simplify by inspection:

19. $\sqrt[3]{a^3-3\,a^2b+3\,ab^2-b^3}$

20. $\sqrt[4]{a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4}$

21. $\sqrt[5]{a^5-5\,a^4b+10\,a^3b^2-10\,a^2b^3+5\,ab^4-b^5}$

EVOLUTION — POLYNOMIALS

To extract the nth root of a polynomial, raise $a+b$ to the nth power, analyze the result to discover how its root may be obtained, and analyze the given expression in a similar manner.

Simplify:

22. $\sqrt{4x^2-12xy+9y^2}$

23. $\sqrt[3]{x^3-6x^2+12x-8}$

24. $\sqrt[4]{x^4-8x^3+24x^2-32x+16}$

25. $\sqrt[5]{x^5-10x^4+40x^3-80x^2+80x-32}$

Ex. 23.

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ &= a^3+(3a^2+3ab+b^2)b\end{aligned}$$

$$\begin{array}{r|l|l}
 & x^3-6x^2+12x-8 & x-2 \\
 & x^3 & a=x \\
\hline
3x^2-6x+4 & -6x^2+12x-8 & b=-2 \\
 & -6x^2+12x-8 & \\
\hline
\end{array}$$

To have a perfect cube before us, we raise $a+b$ to the third power and factor the terms containing b.

If we extract the cube root of a^3, it gives a, the first term of the root; hence, if we extract the cube root of x^3, we must get the first term of the root in this example; the cube root of $x^3=x$, or $a=x$.

If we divide $3a^2b$ by $3a^2$, it gives b, the second term of the root; hence, if we divide $-6x^2$ by what corresponds to $3a^2$, we must get the second term of the root in this example; $3a^2=3x^2$; $-6x^2\div 3x^2=-2$, or $b=-2$.

If we multiply what is within the parenthesis by b, it gives the rest of the power; hence, if we multiply what corresponds to what is within the parenthesis by what corresponds to b, we should get the rest of the power in this example; $3ab=-6x$, $b^2=4$, the parenthesis $=3x^2-6x+4$; multiplying by b, or -2, we get the rest; hence, the root is $x-2$.

NOTE. When the root consists of one or two terms, it may be obtained by inspection as on the preceding page.

If the root consists of more than two terms, the first two may be considered as one term, the first three as one term, and so on.

Simplify:

26. $\sqrt{4x^4-4x^3+5x^2-2x+1}$

27. $\sqrt[3]{27x^6+54x^5+63x^4+44x^3+21x^2+6x+1}$.

28. $\sqrt[3]{x^6-9x^5+33x^4-63x^3+66x^2-36x+8}$

29. $\sqrt{121x^6-66x^5y+119x^4y^2+168x^3y^3-29x^2y^4+90xy^5+81y^6}$

30. $\sqrt[4]{x^8-4x^7+10x^6-16x^5+19x^4-16x^3+10x^2-4x+1}$

Ex. 27.

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ &= a^3+(3a^2+3ab+b^2)b\end{aligned}$$

$$\begin{array}{r|l|l}
 & 27x^6+54x^5+63x^4+44x^3+21x^2+6x+1 & \underline{3x^2+2x+1} \\
 & 27x^6 & \\
27x^4+18x^3+4x^2 & 54x^5+63x^4+44x^3 & a=3x^2 \\
 & 54x^5+36x^4+\ \ 8x^3 & b=2x \\
27x^4+36x^3+12x^2 & 27x^4+36x^3+21x^2+6x+1 & \overline{a}=3x^2+2x \\
9x^2+6x+1 & & b=1 \\
\hline
27x^4+36x^3+21x^2+6x+1 & 27x^4+36x^3+21x^2+6x+1 &
\end{array}$$

Proceeding as before, we find the remainder to be $27x^4+36x^3+\cdots$, and we may regard the first term of the root as the expression, $3x^2+2x$.

If we divide $3a^2b$ by $3a^2$, it gives b, the second term of the root; hence, if we divide $27x^4$ by what corresponds to $3a^2$, we must get the second term of the root in this example; $3a^2=27x^4+36x^3+12x^2$; $27x^4\div 27x^4+\cdots=1$, or b, $=1$.

31. Extract the 5th root of $x^{10}-5x^9+15x^8-30x^7+45x^6-51x^5+45x^4-30x^3+15x^2-5x+1$.

32. Can you obtain the 5th root by extracting the square root of the cube root? What root would this process give?

EVOLUTION — INTEGERS

The nth root of an integer is extracted in the same manner as the nth root of a polynomial. The principles on which the pointing off into periods depends are given in the next two propositions.

Proposition LIV. Theorem

If an integer is separated into periods of n figures each, beginning with units' place, the number of periods will be equal to the number of figures in its nth root.

Let $u =$ a number of one digit; $10t + u$, a number of two digits; $100h + 10t + u$, a number of three digits; and so on.

Since $u < 10$, $u^n < 10^n$, or $u^n < 1$ with n ciphers. No integer less than 1 with n ciphers can consist of more than n figures. Therefore, the nth power of the largest integer of 1 figure cannot consist of more than n figures. Since $10t + u < 100$, $(10t + u)^n < 100^n$, or < 1 with $2n$ ciphers. No integer less than 1 with $2n$ ciphers can consist of more than $2n$ figures. Therefore, the nth power of the largest integer of 2 figures cannot consist of more than $2n$ figures. In a similar way, it may be shown that the nth power of the largest integer of 3 figures cannot consist of more than $3n$ figures, and so on. Hence, the nth power of the largest number of m figures cannot consist of more than mn figures.

Since the smallest integer of m figures is 1 with $m - 1$ ciphers, and since the nth power of 1 with $m - 1$ ciphers is 1 with $(m - 1)n$ ciphers, the nth power of the smallest integer that can be expressed by m figures consists of $(m - 1)n + 1$ figures.

Since n figures make a period, the nth power of the largest integer of m figures consists of $mn \div n$, or m periods; the nth power of the smallest integer of m figures consists of the expression, $(m - 1)n + 1, \div n$, or of $m - 1$ periods of n figures each and of 1 period of 1 figure, or of m periods.

Hence, the principle, Q.E.D.

Proposition LV. Theorem

If an integer is separated into periods of n figures each, beginning at units' place, the nth root of the number denoted by the left-hand period will give the first figure of the root; the nth root of the number denoted by the first two periods will give the first two figures of the root; and so on.

Let $10t+u$ equal a root of two figures, and a the left-hand period when the power is separated into periods of n figures each.

Then, $(10t+u)^n$, or

$$(10t)^n + n(10t)^{n-1}u + \frac{n(n-1)}{2}(10t)^{n-2}u^2 + \cdots = \text{the power} \qquad A$$

To prove that the integral part of $\sqrt[n]{a}$ is t.

$$a = \text{int. part of } t^n + nt^{n-1}\left(\frac{u}{10}\right) + \frac{n(n-1)t^{n-2}}{2}\left(\frac{u}{10}\right)^2 + \cdots \qquad B$$

(If a power is divided by 10^n, the left-hand period will be the integral part of the result. $A \div 10^n$ gives B.)

If $u=0$, $a=t^n$, or $\sqrt[n]{a}=t$; if $u>0$, a may be $>t^n$; if we can prove that $\sqrt[n]{a}$ must be $<t+1$, it follows that the integral part of $\sqrt[n]{a}=t$.

$$(t+1)^n = t^n + nt^{n-1}(1) + \frac{n(n-1)t^{n-2}}{2}(1)^2 + \cdots \qquad C$$

Comparing B and C, term with term, we see that a must always be $<(t+1)^n$, or that $\sqrt[n]{a}$ must always be $<t+1$, because $u \div 10$ must always be <1. Therefore, the principle is true for an integer of two periods.

If the integer consists of three periods, the first two from the left may be regarded as one period, and the whole as an integer of two periods. In this case, the nth root of the two periods combined will give the first two figures of the root. Hence, the principle is true for an integer of three periods. In a similar manner, the truth may be established for any integer.

Hence, the principle, Q.E.D.

EVOLUTION — INTEGERS

To extract the nth root of any integer, point off into periods of n figures each beginning with units' place, extract the root of the number denoted by the first two periods, then the root of the number denoted by the first three periods, and so proceed.

33. Extract the fourth root of 28'4739'6321

$$\begin{aligned}(a+b)^4 &= a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4\\ &= a^4 + (4\,a^3 + 6\,a^2b + 4\,ab^2 + b^3)\,b\end{aligned}$$

	28'4739'6321	231
	16	
32000	12 4739	
7200		$a = 20$
720		$b = 3$
27		$a = 230$
39947	11 9841	$b = 1$
48668000	4898 6321	
317400		
920		
1		
48986321	4898 6321	

We separate the integer into periods of four figures each because the fourth root of the number denoted by the first two periods will give the first two figures of the root, the fourth root of the number denoted by the first three periods will give the first three figures of the root, and so on. Prop. LV.

We extract the fourth root of 28'4739 to obtain the first two figures of the root.

If we extract the 4th root of a^4, it gives a, the first term of the root; hence, if we extract the fourth root of 28, we must get the first figure of the root in this example; the 4th root of $28 = 2+$, or the first figure is 2; $a = 20$ because 28 is not 28 units but 28 ten-thousands.

If we divide $4\,a^3b$ by $4\,a^3$, it gives b, the second term of the root; hence, if we divide what corresponds to $4\,a^3b$, or the greater part of 124739, by what

corresponds to $4a^3$, we must get the second figure of the root in this example; $4a^3 = 32000$; $124739 \div 32000 = 3 +$, or $b = 3$.

If we multiply what is within the parenthesis by b, it gives the rest of the power; hence, if we multiply what corresponds to what is within the parenthesis by what corresponds to b, we should get the rest of the power in this example if it is a perfect power; $4a^3 = 32000$; $6a^2b = 7200$, $4ab^2 = 720$, $b^3 = 27$, the parenthesis $= 39947$; multiplying by b, or 3, and subtracting, we get 4898. Hence, the 4th root of 284739 is $23 +$.

We extract the 4th root of 284739′6321 to obtain the first three figures of the root. We may regard the whole as 284739 ten-thousands and 6321 units.

If we extract the 4th root of a^4, it gives a, the first term of the root; hence, if we extract the 4th root of 284739, we must get the first part (the number denoted by the first two figures) of the root in this example; the 4th root of 284739 is $23 +$, or the number denoted by the first two figures of the root is 23; $a = 230$ because 284739 is not 284739 units but 284739 ten-thousands.

If we divide $4a^3b$ by $4a^3$, it gives b, the second term of the root; hence, if we divide what corresponds to $4a^3b$, or the greater part of 48986321, by what corresponds to $4a^3$, we must get the second part (third figure) of the root in this example; $4a^3 = 48668000$; $48986321 \div 48668000 = 1 +$, or $b = 1$.

If we multiply what is within the parenthesis by b, it gives the rest of the power; hence, if we multiply what corresponds to what is within the parenthesis by what corresponds to b, we should get the rest of the power in this example if it is a perfect power; $4a^3 = 48668000$; $6a^2b = 317400$, $4ab^2 = 920$, $b^3 = 1$, the parenthesis $= 48986321$; multiplying by b, or 1, and subtracting, we get 0. Hence, the 4th root of 2847396321 is 231.

34. What are the steps in the extraction of the cube root of 340′068′392?

First, the cube root of 340; second, the cube root of 340′068; third, the cube root of 340′068′392.

Simplify:

35. $\sqrt[3]{340068392}$

36. $\sqrt[4]{1642047467776}$

37. $\sqrt[5]{657748550151}$

38. $\sqrt[6]{7529536}$

39. $\sqrt[10]{61917364224}$

40. $\sqrt[2]{61917364224}$

EVOLUTION — FRACTIONS

It has been shown that the nth root of a fraction is the nth root of the numerator divided by the nth root of the denominator. If the denominator is not a perfect power, both terms may be multiplied or divided by any number that will fulfill this condition, or the fraction may be reduced to a decimal.

41. What is the $\sqrt{\frac{4}{9}}$? $\sqrt[3]{\frac{8}{27}}$? $\sqrt[4]{\frac{16}{81}}$?

42. In terms of the root of a fraction whose denominator is a perfect power corresponding to the index of the root, what is the equivalent of $\sqrt{\frac{2}{3}}$? $\sqrt[3]{\frac{2}{3}}$? $\sqrt[4]{\frac{2}{3}}$?

PROPOSITION LVI. THEOREM

In the extraction of the nth root, a decimal may be pointed off into periods of n figures each, beginning at the decimal point. There will be as many decimal places in the root as there are decimal periods in the power.

Since a decimal is a common fraction, the preparation for the extraction of its nth root consists in multiplying both terms by 1 with enough ciphers to make the denominator a perfect nth power.

Since the number of ciphers in the new denominator must be a multiple of n, the number of decimal places in the equivalent decimal must be the same multiple of n.

(The number of decimal places in a decimal is the same as the number of ciphers in its denominator.)

It follows that annexing ciphers to a decimal until its denominator is a perfect power and pointing off the numerator from units, must give the same result as pointing off the decimal from the decimal point and annexing sufficient ciphers to complete the right-hand period.

Hence, the principle, Q.E.D.

43. What is the numerator of .003265? the denominator? How many decimal places are there? How many ciphers in the denominator?

44. If the numerator 3265 is pointed off from units into periods of 3 figures each, what are the periods?

45. If the decimal .003265 is pointed off from the decimal point into periods of 3 figures each, what are the periods?

46. Is the denominator of .003265 a perfect 4th power?

47. If the 4th root of .003265 is to be extracted, the equivalent decimal is .00326500. Show that the decimal may be pointed off from the decimal point or that the numerator may be pointed off from units.

48. The figures in the cube root of .000′002′744 are 14. How many decimal places in the root? Point off the root.

49. In terms of the corresponding root of a decimal, what is the equivalent of $\sqrt{\frac{2}{5}}$? $\sqrt[3]{\frac{2}{3}}$? $\sqrt[4]{\frac{3}{4}}$?

50. Extract the square root of $\frac{2}{8}$; extract the square root of each term and simplify.

51. Extract the square root of $\frac{2}{3}$; in preparation, multiply both terms by 3.

52. Extract the square root of $\frac{2}{8}$; in preparation, reduce to a decimal.

Simplify:

53. $\sqrt{\frac{24336}{404496}}$

54. $\sqrt[3]{\frac{2662}{16}}$

55. $\sqrt[4]{\frac{625}{4}}$

56. $\sqrt[3]{.056623104}$

57. $\sqrt[5]{9161.32832}$

58. $\sqrt[3]{2}$ to 4 places

LOGARITHMS — COMMON SYSTEM

In the common system, the logarithm of a number is the power to which 10 must be raised to equal that number.

Any number may be taken as the base of a system of logarithms. Since $3^4 = 81$, 4 is the logarithm of 81 in the system whose base is 3; since $9^2 = 81$, 2 is the logarithm of 81 in the system whose base is 9; since $10^2 = 100$, 2 is the logarithm of 100 in the system whose base is 10. These relations may be written $\log_3 81 = 4$, $\log_9 81 = 2$, $\log_{10} 100 = 2$, but when the base is 10 it is customary to omit it both in writing and in reading; $\log 100 = 2$ means $\log_{10} 100 = 2$.

Logarithms afford a method of multiplying, dividing, raising to powers, and depressing to roots.

1. Find the value of 4×8 by logarithms.

$4 = 2^2$; $8 = 2^3$; $4 \times 8 = 2^{2+3} = 2^5$; since $32 = 2^5$, $4 \times 8 = 32$. Or, $\log_2 4 = 2$; $\log_2 8 = 3$; $\log_2 (4 \times 8) = 2 + 3 = 5$; since 32 is the number whose $\log_2$ is 5, $4 \times 8 = 32$.

2. Find the value of $32 \div 4$ by logarithms.

$32 = 2^5$; $4 = 2^2$; $32 \div 4 = 2^{5-2} = 2^3$; since $8 = 2^3$, $32 \div 4 = 8$. Or, $\log_2 32 = 5$; $\log_2 4 = 2$; $\log_2 (32 \div 4) = 5 - 2 = 3$; since 8 is the number whose $\log_2$ is 3, $32 \div 4 = 8$.

3. Find the value of 8^2 by logarithms.

$8 = 2^3$; $8^2 = 2^{2 \times 3} = 2^6$; since $64 = 2^6$, $8^2 = 64$. Or, $\log_2 8 = 3$; $\log_2 8^2 = 2 \cdot \log_2 8 = 2 \times 3 = 6$; since 64 is the number whose $\log_2$ is 6, $8^2 = 64$.

4. Find the value of $\sqrt[3]{64}$ by logarithms.

$64 = 2^6$; $\sqrt[3]{64} = 2^{6 \div 3} = 2^2$; since $4 = 2^2$, $\sqrt[3]{64} = 4$. Or, $\log_2 64 = 6$; $\log_2 \sqrt[3]{64} = \log_2 64 \div 3 = 6 \div 3 = 2$; since 4 is the number whose $\log_2$ is 2, $\sqrt[3]{64} = 4$.

NOTE. Nothing can be done by means of logarithms that cannot also be done by multiplying, dividing, raising to powers, and depressing to roots, but the use of logarithms often saves much labor.

The logarithm of every positive number that is not an integral power of 10 consists of an integral part called the characteristic and a decimal part called the mantissa. The mantissa is always positive.

$$10^0 = 1, \qquad 10^1 = 10, \qquad 10^2 = 100, \qquad 10^3 = 1000$$

Since $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, and so on, the logarithm of every number between 1 and 10 is between 0 and 1, or 0+; the logarithm of every number between 10 and 100 is between 1 and 2, or 1+; the logarithm of every number between 100 and 1000 is between 2 and 3, or 2+; and so on.

5. Between what integers must log 9834 be found?

Ans. Between 3 and 4. 9834 is between 1000 and 10000; $\log 1000 = 3$; $\log 10000 = 4$; $\therefore \log 9834 = 3+$.

The characteristic of the logarithm of a number is found by the next two propositions; the mantissa is found by reference to tables. Vega's seven-place tables consisting of one hundred eighty-six pages are satisfactory for most computations. Four-place tables (p. 148) can be printed on two pages; they are satisfactory for illustrating the uses of logarithms and for computations in which the results are true to three or four orders. The method of computing the mantissas which form the tables is not explained in this book.

The number which corresponds to a given logarithm is called its antilogarithm.

The cologarithm of a number is zero minus its logarithm.

Thus, $\log 100 = 2$; antilog $2 = 100$. $\log 100 = 2$; colog $100 = -2$.

6. What is the number whose logarithm is 5? In other words, what is the antilog 5?

Ans. 100000. Since $100000 = 10^5$, antilog $5 = 100000$.

7. What is colog 1000?

Ans. -3. $\log 1000 = 3$; colog $1000 = 0 - 3$, or -3.

LOGARITHMS — CHARACTERISTICS

PROPOSITION LVII. THEOREM

The characteristic of the logarithm of an integer or mixed decimal is positive and one less than the number of integral digits.

Since $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, and so on, the characteristic of the logarithm of every number between 1 and 10 is 0, or one less than the number of integral digits; between 10 and 100, 1, or one less than the number of integral digits; between 100 and 1000, 2, or one less than the number of integral digits; and so on.

Hence, the principle, Q.E.D.

Find the characteristic of the logarithm:

8. 568	**11.** 3785	**14.** 25
9. 56.8	**12.** 378.5	**15.** 250
10. 5.68	**13.** 37.85	**16.** 2500

17. If the figures of the antilogarithm are 23645, and the characteristic of the logarithm is 0, what is the antilogarithm?

Ans. 2.3645. Since the characteristic is 0, the number of integral digits in the antilogarithm must be one more than 0, or 1.

18. In Ex. 17, what is the antilogarithm if the characteristic is 3?

Point off the result when the figures of the antilogarithm are 23645 *and the characteristic:*

19. 6; 2	**20.** 8; 1	**21.** 4; 5

Ex. 19. 2364500. Since the characteristic is 6, the number of integral digits in the antilogarithm must be one more than 6, or 7.

PROPOSITION LVIII. THEOREM

The characteristic of the logarithm of a decimal is negative and the same as the number of the place occupied by the first significant figure.

$$.1 = \frac{1}{10} = 10^{-1} \qquad .001 = \frac{1}{10^3} = 10^{-3}$$

$$.01 = \frac{1}{10^2} = 10^{-2} \qquad .0001 = \frac{1}{10^4} = 10^{-4}$$

Log .1 = − 1, log .01 = − 2, log .001 = − 3, and so on.

Since every decimal whose first integral digit is in tenths' place is .1 or a number between .1 and 1, and since the log .1 = −1 and log 1 = 0, the log of such a decimal is − 1, or is a negative number between − 1 and 0. That is, it must be − 1 plus a positive decimal.

Since every decimal whose first integral digit is in hundredths' place is .01 or a number between .01 and .1, and since log .01 = −2 and log .1 = − 1, the log of such a decimal is − 2, or is a negative number between − 2 and − 1. That is, it must be − 2 plus a positive decimal.

In the same way it may be shown that the characteristic of every decimal whose first integral digit is in any decimal place is negative and the same as the number of the place occupied by its first significant figure.

Hence, the principle, Q.E.D.

NOTE. As a characteristic, − 1 is written 9 − 10; − 2, 8 − 10; − 3, 7 − 10, ... The characteristic of a logarithm may be positive or negative, but the mantissa is always positive. − 2.3864 as a logarithm does not have the signification which it has elsewhere of −2−.3864, but means −2+.3864.

Find the characteristic of the logarithm:

22. .25	**25.** .123	**28.** .02345
23. .025	**26.** .0123	**29.** .002345
24. .0025	**27.** .00123	**30.** .0002345

LOGARITHMS — MANTISSAS

PROPOSITION LIX. THEOREM

Moving the decimal point of a number in either direction does not affect the mantissa of the logarithm.

To move the decimal point is to multiply by 10 to a positive integral or by 10 to a negative integral power.

Multiplying by 10 to a positive integral power increases the logarithm by an integer.

Multiplying by 10 to a negative integral power decreases the logarithm by an integer.

In either case the mantissa is not affected.

Hence, the principle, Q.E.D.

In a four-place table, the first two figures of numbers are placed at the left of the page in a vertical column headed No.; the third figure, in a horizontal line at the top. The mantissa of a number of three figures is placed at the intersection of a horizontal line running through the first two figures, and a vertical line through the third; the mantissa of a number of four or five figures is found by interpolation; the mantissa of a number of more than five figures is the same as for five figures. The decimal point is not printed.

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7498	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627

31. Find the mantissa of log 550; of log 561; of log 564; of log 565.

Mantissa of log 550 = .7404; mantissa of log 561 = .7490; mantissa of log 564 = .7513; mantissa of log 565 = .7520.

To find the mantissa of the logarithm of a number of more than three figures, find the mantissa of the logarithm of the number denoted by its first three figures, and add the sum of the results found by multiplying its fourth figure by the tabular difference and dividing by 10 and multiplying its fifth figure by the tabular difference and dividing by 100. *Do the work mentally.*

Let us find the log of 564.28.

Since 564.28 lies between 564 and 565, its mantissa must lie between the mantissa of 564 and the mantissa of 565, or between 7513 and 7520. The difference between 564 and 565 is 1; between 7513 and 7520 is 7. Since a difference of 1 in the number makes a difference of 7 in the mantissa, a difference of .28 in the number will make a difference of .28 of 7, or of 1.96 in the mantissa.

Therefore, the mantissa of 564.28 is 1.96 greater than the mantissa of 564. We consider the integral part only, but count .5 or more as 1, and neglect .4 or less. Adding 2, the mantissa of log 564.28 is .7515.

Hence, to get the correction, we multiply the number denoted by the fourth and the fifth figures by the tabular difference, and divide by 100. Or, since .28 of 7 = .2 of 7 + .08 of 7, we multiply the fourth figure by the tabular difference and divide by 10; we multiply the fifth figure by the tabular difference and divide by 100; and we add the results.

In practice, we hold a finger on 7513 until the mantissa is found. We say, 7 (tabular difference), 1.4 ($2 \times 7 \div 10$), .5 ($8 \times 7 \div 100$), 1.9 ($1.4 + .5$), 2 (amount to add), 7515 ($7513 + 2$).

Find:

32. log 896.425	**36.** log .7	**40.** log .968327
33. log 3.2467	**37.** log .07	**41.** log .0034969
34. log .0093278	**38.** log .86543	**42.** log .0003396
35. log .035007	**39.** log .086543	**43.** log .0030064

Ex. 32. We hold a finger on 9523 until the mantissa is found. We say, 5, 2, .1, 2.1, 2, 9525, 2.9525 (characteristic and mantissa).

LOGARITHMS — MANTISSAS

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

LOGARITHMS — MANTISSAS

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

LOGARITHMS — ANTILOGARITHMS

To find the antilogarithm, find in the table the mantissa nearest the given mantissa, but smaller. This will give the first two figures in the column headed No., and the third figure at the top of the page. For the fourth and fifth figures, subtract the mantissa thus found from the given mantissa, multiply by 100, and divide by the tabular difference. Point off as the characteristic indicates. *Do the work mentally.*

Let us find the antilogarithm of 2.7515.

Finding the antilogarithm is the reverse of finding the logarithm. The last thing in finding the logarithm was to add something to the mantissa in the table; in the reverse process, we must find the nearest smaller mantissa to 7515 in the table, and find what has been added. The nearest smaller mantissa is 7513; the difference, 2.

Before this, in finding the logarithm, we multiplied by the tabular difference and divided by 100; in the reverse process, we must multiply by 100 and divide by the tabular difference; $100 \times 2 = 200$; $200 \div 7 = 28+$; the fourth and fifth figures are 28.

The third figure is 4, in the vertical line over 7513, and the first two figures are 56, in the horizontal line at the left.

Since the characteristic is 2, there must be three figures in the integral part of the result. Therefore, antilog $2.7515 = 564.28$.

Find x when log:

44. $x = 2.9525$	**48.** $x = 9.8451 - 10$	**52.** $x = 9.9860 - 10$
45. $x = 0.5114$	**49.** $x = 8.8451 - 10$	**53.** $x = 7.5437 - 10$
46. $x = 9.9698 - 10$	**50.** $x = 9.9372 - 10$	**54.** $x = 6.5310 - 10$
47. $x = 8.5442 - 10$	**51.** $x = 8.9372 - 10$	**55.** $x = 7.4781 - 10$

Ex. 46. We hold a finger on 9694 until the antilogarithm is found. We say, 4 (difference of mantissas), 400 (4×100), 5 (tabular difference), 80 ($400 \div 5$), 93280 (the sequence of digits), .9328 (antilogarithm). Since the characteristic is -1, the first significant digit must be in the first decimal place.

LOGARITHMS — COLOGARITHMS

To find the cologarithm of a number, find the mantissa as in logarithms; find what must be added to each digit from the left to make 9, but to the last integral digit to make 10; find what must be added to the characteristic to make 9. If the number is an integer, annex -10. *Do the work mentally.*

Let us find colog 564.28. Let us find colog .056428.

Since the colog $564.28 = 0 - \log 564.28$ by definition, we must find log 564.28 and subtract from 0.

$$\begin{array}{rl} 0 = 10 - 10 = & 9.999\,(10) - 10 \\ \log 564.28 = & \underline{2.751\ \ 5\qquad\quad} \\ \text{colog } 564.28 = & 7.248\ \ 5\ \ - 10 \end{array} \qquad \begin{array}{rl} 0 = 10 - 10 = & 9.999\,(10) - 10 \\ \log .056428 = & \underline{8.751\ \ 5\ \ - 10} \\ \text{colog } .056428 = & 1.248\ \ 5 \end{array}$$

In finding colog 564.28, we see that 2 is what must be added to 7 to make 9; 4 is what must be added to 5 to make 9; 8 is what must be added to 1 to make 9; 5 is what must be added to 5 to make 10. 7 is what must be added to the characteristic to make 9. The number, 564, is an integer, and -10 is annexed.

In finding colog .056428, we see that if the number is a decimal, -10 must *not* be annexed.

Find:

56. colog 48.136 **60.** colog 1.2456 **64.** colog 7324.69

57. colog 70.329 **61.** colog 36.549 **65.** colog .867003

58. colog 8.3456 **62.** colog 1 **66.** colog .928563

59. colog .38469 **63.** colog .37854 **67.** colog 743.691

Ex. 56. We hold a finger on 6821 until the cologarithm is found. We say, 9, 2.7, .5, 3.2, 3, 6824, 3176 (each figure of mantissa from 9 but last integral figure from 10), 1 (characteristic of logarithm), 8.3176–10 (cologarithm).

LOGARITHMS — THE OPERATIONS

PROPOSITION LX. THEOREM

The logarithm of the product of two quantities is the logarithm of the multiplicand plus the logarithm of the multiplier.

To prove $\log ab = \log a + \log b$.

Let $a = 10^x$

and $b = 10^y$

then must $ab = 10^{x+y}$

$$\log a = x; \quad \log b = y; \quad \log ab = x + y \qquad (1)$$

(The logarithm of a number is the power to which 10 must be raised to equal that number.)

Substituting in (1), $\log ab = \log a + \log b$

Hence, the principle, Q.E.D.

PROPOSITION LXI. THEOREM

The logarithm of the quotient of two quantities is the logarithm of the dividend plus the cologarithm of the divisor.

To prove $\log \frac{a}{b} = \log a + \text{colog}\, b$.

Let $a = 10^x$

and $b = 10^y$

then must $\frac{a}{b} = 10^{x-y}$

$$\log a = x; \quad \log b = y; \quad \log \frac{a}{b} = x - y \qquad (1)$$

(The logarithm of a number is the power to which 10 must be raised to equal that number.)

Substituting in (1), $\log \frac{a}{b} = \log a - \log b = \log a + \text{colog}\, b$

Hence, the principle, Q.E.D.

PROPOSITION LXII. THEOREM

The logarithm of a number raised to any power is equal to the product of the exponent by the logarithm of the number.

Let us take a^n

To prove $\log a^n = n \log a$

$$a^n = aaaa \cdots, \text{ or } a, \ n \text{ times}$$

$$\log a^n = \log a + \log a + \cdots, \text{ or } \log a, \ n \text{ times.}$$

(The logarithm of the product of two or more quantities is the sum of the logarithms of the quantities.)

$$\therefore \ \log a^n = n \log a$$

Hence, the principle, Q.E.D.

Reduce to logarithmic form:

68. 36.98×4.375

69. $26843 \times .38569$

70. $17.846 \times .0036947$

71. $.038065 \times .039825$

72. $.72345 \div 92.834$

73. $24789 \div .017846$

74. $.034069 \div .013758$

75. $632.96 \div .369481$

76. $(363.948)^{\frac{3}{4}}$

77. $(.73296)^{\frac{2}{5}}$

78. $\sqrt[4]{.086345}$

79. $\sqrt[3]{.004839^2}$

80. $\dfrac{bc^2}{dm^2}$; $\dfrac{a^2b^2}{c^2d^3}$

81. $\dfrac{a^2c}{b^3d}$; $\dfrac{a^3b^3}{cd}$

82. $\sqrt[4]{\dfrac{a^5bx^3}{c^5m^3x}}$; $\sqrt[3]{\dfrac{a^2b}{c^2d^2}}$

83. $\sqrt{\dfrac{m^2n^2x}{x^3y^2z}}$; $\sqrt[5]{\dfrac{a^4}{b^2c^3}}$

Ex. 68. $\log (\quad) = \log 36.98 + \log 4.375$

Ex. 72. $\log (\quad) = \log .72345 + \text{colog } 92.834$

Ex. 76. $\log (\quad) = \frac{3}{4} \log 363.948$

Ex. 82. $\log (\quad) = \frac{1}{4}(5 \log a + \log b + 3 \log x + 5 \text{ colog } c + 3 \text{ colog } m + \text{colog } x)$

LOGARITHMS — EXAMPLES

84. By means of logs find the value in Ex. 68.

85. By means of logs find the value in Ex. 72.

Ex. 84. 36.98×4.375

$$\log(\ ) = \log 36.98 + \log 4.375$$

$$\begin{aligned} \log 36.98 &= 1.5680 \\ \log 4.375 &= \underline{\ .6410} \\ \log(\ ) &= 2.2090 \\ (\ ) &= 161.81 \end{aligned}$$

Ex. 85. $.72345 \div 92.834$

$$\log(\ ) = \log .72345 + \text{colog } 92.834$$

$$\begin{aligned} \log .72345 &= 9.8594 - 10 \\ \text{colog } 92.834 &= \underline{8.0323 - 10} \\ \log(\ ) &= 7.8917 - 10 \\ (\ ) &= .0077933 \end{aligned}$$

86. By means of logs find the value in Ex. 78.

87. By means of logs find the value in Ex. 77.

Ex. 86. $\sqrt[4]{.086345}$

$$\begin{aligned} \log(\ ) &= \tfrac{1}{4} \log .086345 \\ \log .086345 &= 8.9362 - 10 \\ \tfrac{1}{4} \log .086345 &= 9.7341 - 10 \\ (\ ) &= .5421 \end{aligned}$$

Ex. 87. $(.73296)^{\frac{2}{5}}$

$$\begin{aligned} \log(\ ) &= \tfrac{2}{5} \log .73296 \\ \log .73296 &= 9.8651 - 10 \\ \tfrac{2}{5} \log .73296 &= 9.9460 - 10 \\ (\ ) &= .8830 \end{aligned}$$

NOTE. In dividing a negative logarithm by an integer, we add enough to both parts of the characteristic to obtain a sum in the second part whose quotient is 10. Thus:

$$4\overline{)8.9362 - 10} \quad (1) \qquad\qquad 9.8651 - 10 \quad (2) \qquad\qquad 5\overline{)9.7302 - 10} \quad (3)$$
$$\ \ 9.7341 - 10 \qquad\qquad\qquad \underline{\qquad 2 \qquad} \qquad\qquad\qquad 9.9460 - 10$$
$$9.7302 - 10$$

In (1), we add 30 to both parts of the characteristic; $8 - 10$ becomes $38 - 40$; we divide $38.9362 - 40$ by 4; the object is to keep the second part of the characteristic -10.

In (2), we multiply and obtain $19.7302 - 20$, but write the result $9.7302 - 10$.

In (3), we add 40 to both parts of the characteristic and divide $49.7302 - 50$ bv 5.

Find the value of:

88. $\dfrac{28.369 \div 327.9145}{.68496}$

89. $\dfrac{.17648 \div .78563}{.62934 \div 13965}$

90. $\dfrac{\sqrt{64869} \div \sqrt{32.568}}{\sqrt{.064321} \times \sqrt{78.564}}$

91. $\sqrt[5]{\dfrac{72.568 \times .324695}{6.8327 \div .00179}}$

92. $\dfrac{.017643 \times 68.596}{32456 \times .72468}$

93. $\dfrac{\sqrt{3.2469} \div \sqrt{65.003}}{\sqrt{6.3264} \times \sqrt{78326}}$

94. $\sqrt[3]{\dfrac{732.056^2 \times .0003572^4 \times 89793}{42.2798^3 \times 3.4574 \times .0026518^5}}$

95. $\sqrt[4]{\dfrac{.0062 \times (.0007)^{\frac{1}{2}} \times 31.257^{\frac{1}{3}}}{3.6^4 \times .00005 \times 9786.432^2}}$

Find x:

96. $3^{x-2} = 5$

97. $5^x = 8^3$

98. $3^{x-4} = 7$

99. $.725^x = .0842$

100. $7.25^x = .842$

101. $a^{px}b^{qx} = r$

102. $ma^{\frac{1}{x}} = n^2$

103. $.0825^x = 16$

Ex. 96. $(x-2)\log 3 = \log 5$; $x - 2 = \dfrac{\log 5}{\log 3} = \dfrac{.6990}{.4771} = 1.465$; $x = 3.465$.

Ex. 99. $x \log .725 = \log .0842$; $x = \dfrac{\log .0842}{\log .725} = \dfrac{-2.9253}{-1.8603} = \dfrac{-2 + .9253}{-1 + .8603} = \dfrac{-1.0747}{-.1397} = 7.69$. Remember that -2.9253 means $-2 + .9253$ (p. 145), where the mantissa is '+' and the characteristic '−.' Before division is possible, the logarithm must be reduced to a form where both mantissa and characteristic have the same sign; $-2 + .9253 = -1 - .0747$, or -1.0747.

COMPOUND INTEREST

PROPOSITION LXIII. THEOREM

The amount of \$1 for n years at r%, compound interest, is $(1+r)^n$ *dollars.*

By definition, the amount at the end of each year bears interest during the next year.

At the end of the first year the amount is $1 + r$ dollars; during the year \$1 amounts to $1 + r$ dollars, and $1 + r$ dollars amounts to $1 + r$ times $1 + r$ dollars, or $(1 + r)^2$ dollars. At the end of the second year the amount is $(1 + r)^2$ dollars; during the year \$1 amounts to $(1 + r)$ dollars, and $(1 + r)^2$ dollars amounts to $(1 + r)^2$ times $(1 + r)$ dollars, or $(1 + r)^3$ dollars. And so on.

Hence, the principle, Q.E.D.

Express the result by factors. Find the amount:

1. Of \$1 for 18 yr. at 6% compound interest.
2. Of \$100 for 18 yr. at 6% compound interest.
3. Of \$750 for 60 yr. at 4% compound interest.
4. Of 1¢ for 60 yr. at 4% compound interest.

If the time is for years, months, and days, the amount of the principal must be found for the years, and the amount of the result for the months and days.

Express the result by factors. Find the amount:

5. Of \$1 for 3 yr. 6 mo. at 6% compound interest.
6. Of \$100 for 3 yr. 6 mo. at 6% compound interest.
7. Of \$100 for 5 yr. 6 mo. 6 da. at 6% compound interest.
8. Of \$125 for 7 yr. 7 mo. 7 da. at 5% compound interest.

Ex. 5. *Ans.* $\$(1.06)^3(1.03)$. An approximate result is

$\$(1.06)^{3\frac{1}{2}}$, or $\$(1.06)^3(1.06)^{\frac{1}{2}}$.

PROPOSITION LXIV. THEOREM

The amount of \$1 for n years at r% interest compounded m times a year is the amount of \$1 for mn years at $\frac{r}{m}$%, interest compounded annually.

$\$\left(1+\frac{r}{m}\right)^{mn}=$ the amount of \$1 for mn years at $\frac{r}{m}\%$, interest compounded annually.

$\$\left(1+\frac{r}{m}\right)^{mn}=$ the amount of \$1 for n years at $r\%$, interest compounded m times a year.

(In n years the interest is compounded mn times; the amount at the end of the first period is $\$\left(1+\frac{r}{m}\right)$; at the end of the second, $\$\left(1+\frac{r}{m}\right)^2$; $\cdots$ at the end of the mnth, $\$\left(1+\frac{r}{m}\right)^{mn}$.

Hence, the principle, Q.E.D.

When interest is compounded oftener than once a year, the example should be reduced to an equivalent example in which the interest is compounded annually.

9. What is the amount of \$100 for 3 yr. 5 mo. 6 da. at 6%, interest compounded quarterly? Do not solve, but state the equivalent example, interest payable annually.

10. What is the amount of \$825.67 for 2 yr. 3 mo. 7 da. at 4%, interest compounded semiannually? Do not solve, but state the equivalent example.

Logarithms should be employed in the following computations.

Find in dollars the answers to:

11. Ex. 1	**14.** Ex. 4	**17.** Ex. 8
12. Ex. 2	**15.** Ex. 6	**18.** Ex. 9
13. Ex. 3	**16.** Ex. 7	**19.** Ex. 10

COMPOUND INTEREST

20. In what time will \$100 amount to \$137.93 at 6% compound interest? Before solving, state your plan of procedure.

First, find the integral part of the time in years; second, find the compound amount of \$100 for this integral number of years at 6%; third, subtract this amount from \$137.93 to find the interest for the fractional part of a year; fourth, find the time in which the amount for the integral number of years will gain this difference at 6% simple interest.

First Step. Let $x =$ number of years; $100(1.06)^x = 137.93$; $x = (\log 137.93 - \log 100) \div .0253 = .1397 \div .0253 = 5.5214$; 5 is the integral part.

Second Step. *Ans.* \$133.81. *Third Step.* *Ans.* \$4.12.

Fourth Step. In what time will \$133.81 gain \$4.12 at 6% simple interest? *Ans.* 6 mo. 5 da.

NOTE. Compare with Ex. 7. There is a discrepancy of 1 da. due to the fact that we have used only a four-place logarithm table.

21. At what per cent compound interest will \$100 amount to \$137.93 in 5 yr. 6 mo. 6 da.? Before solving, state your plan.

Since the amount at $x\%$ may be approximately obtained (Ex. 5) by regarding n a mixed number of years,

$$100\left(1 + \frac{x}{100}\right)^{5\frac{31}{60}} = 137.93$$

approximately. We will find x by solving this equation. We will then take the nearest integer and see if it will answer the conditions.

22. What difficulty arises in the attempt to solve Ex. 21 as follows? Let $x =$ the rate; $100\left(1 + \frac{x}{100}\right)^5\left(1 + \frac{x}{100} \cdot \frac{31}{60}\right) = 137.93$.

This is the natural way to solve the example, but the equation which results is of the fifth degree with x in more than one term and is, therefore, very difficult of solution.

23. What principal at 6% compound interest will amount to \$137.93 in 5 yr. 6 mo. 6 da.? State your plan.

We will let x equal the principal. Then,

$$x(1.06)^5 \times 1.031 = 137.93.$$

From this equation, x may readily be found.

24. If the amount of \$$m$ at compound interest for 7 yr. is \$236.58 at 1%, what is the amount at 6%?

$$m(1.01)^7 = 236.58;\ m = 220.70$$

$$220.70(1.06)^7 = 331.84$$

25. Why will not 6 times \$236.58 give the correct result in Ex. 24?

26. In Ex. 24, by what should \$236.58 be multiplied?

Ex. 26. $\dfrac{m(1.06)^7}{m(1.01)^7} = \left(\dfrac{1.06}{1.01}\right)^7$; $\log(\ ) = 7[\log 1.06 + \operatorname{colog} 1.01]$; $= .1470$; $(\ ) = 1.4029$. Multiply by 1.4029.

27. What is the amount at simple interest of 1¢ for 1902 yr. at 6%?

28. What is the amount at compound interest of 1¢ from the year 1 to the year 1903 at 6%?

29. What is the value in dollars of a sphere of pure gold whose radius is the distance from the earth to the sun? Data: volume of a sphere, $\frac{4}{3}\pi r^3$; distance from the earth to the sun, 92,000,000 mi.; weight of 1 cu. ft. of water, $62\frac{1}{2}$ lb. avoirdupois; specific gravity of gold, 19.2; number of grains of pure gold in a dollar, 23.2.

30. How many spheres of pure gold, the radius of each being the distance from the earth to the sun, are required to pay the compound interest at 6%, of 1¢ from the year 1 to the year 1903?

NEGATIVE AND FRACTIONAL EXPONENTS

DEVELOPMENT

All of the principles that have been developed for positive integral exponents are also true for negative and fractional exponents. Expressions involving negative and fractional exponents may be added and subtracted, multiplied and divided, factored, treated in equations, and used in every way that expressions involving positive integral exponents may be used. It remains to prove that the principles already developed apply to negative and fractional exponents and to consider special cases.

A more complete explanation may also be given of the outgrowth from the agreement that '+' and '−' shall denote movements in opposite directions. See p. 10, Prop. II.

ADDITION AND SUBTRACTION. If a quantity increased by b equals a, the quantity must be a diminished by b. This statement is a self-evident proposition. It has been agreed to represent "*use as an addend*" by '+' written before the base; then the opposite, or "*use as a subtrahend*," must be expressed by '−' similarly placed. The statement may then be abbreviated:

$$\text{If}\quad x + b = a,\ x = a - b$$

MULTIPLICATION AND DIVISION. If a quantity multiplied by b equals a, the quantity must be a divided by b. This statement

is a self-evident proposition. It has been agreed to represent "*multiplied*" by '×,' and "*divided*" by '÷,' or by the fractional form. The statement may then be abbreviated:

$$\text{If} \quad x \times b = a, \quad x = a \div b$$

or,

$$\text{If} \quad x \times b = a, \quad x = \frac{a}{b}$$

or,

$$\text{If} \quad x \times b = a, \quad x = a \times b^{-1}$$

See note below

INVOLUTION, EVOLUTION, AND LOGARITHMS. If a quantity multiplied by b, the result multiplied by b, and so on n times, equals a, the quantity must be a divided by b, the result divided by b, and so on n times. This statement is a self-evident proposition. It has been agreed to represent "*use as a multiplier*" by '+' written above the base and to the left of the exponent; then the opposite, or "*use as a divisor*," must be expressed by '−' similarly placed. The statement may then be abbreviated:

$$\text{If} \quad x \times b^n = a, \quad x = a \times b^{-n}$$

or,

$$\text{If} \quad x \times b^n = a, \quad x = \frac{a}{b^n}$$

or,

$$\text{If} \quad x \times b^n = a, \quad x = a \div b^n$$

NOTE. If $n = 1$, $x \times b^{+n} = a$, becomes $x \times b = a$, and $x = a \times b^{-1}$.

If the nth power of a number is a, the number must be the nth root of a. If b raised to any power is equal to a, the exponent of the power is the logarithm of a in the system whose base is b. The first statement is a self-evident proposition; the second is true by definition. The statements may be abbreviated:

$$\text{If} \quad x^n = a, \quad x = \sqrt[n]{a}$$

or,

$$\text{If} \quad x^n = a, \quad x = a^{\frac{1}{n}}$$

See p. 165

or,

$$\text{If} \quad x^n = a, \quad x = a^{(n^{-1})}$$

or,

$$\text{If} \quad b^x = a, \quad x = \log_b a$$

NEGATIVE EXPONENTS

Proposition VI. Agreement (p. 14)

A positive exponent shows how many times the base is used as a multiplier.

Thus, $a^{+n} = a \times a \times a \cdots$, or $aaaa \cdots$, where a is used n times as a multiplier.

Proposition VII. Theorem (p. 14)

A negative exponent shows how many times the base is used as a divisor.

To prove that $a^{-n} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \cdots$, or $\frac{1}{a \times a \times a \cdots}$, or $\frac{1}{a^n}$.

a^{+n} means that a is used n times as a multiplier. Since '$-$' means the opposite of '$+$,' a^{-n} means that a is used n times in a sense opposite to a multiplier, or n times as a divisor.

Hence, the principle, Q.E.D.

Proposition VIII. Theorem (p. 14)

A zero exponent shows that the base is used the same number of times both as a multiplier and as a divisor; the value of a number with zero exponent is 1.

$$0 = +n - n$$

$$\therefore\ a^0 = a^{+n-n} = \frac{a \times a \times a \cdots}{a \times a \times a \cdots} = 1$$

(In a^{+n}, a is used n times as a multiplier; in a^{-n}, a is used n times as a divisor.)

Hence, the principle, Q.E.D.

Note. The pupil should compare these demonstrations with those of p. 14. They are given here in general form.

PROPOSITION LXV. THEOREM

To transfer a factor from one term of a fraction to the other, change the sign of its exponent.

To prove that $$\frac{a^n}{1}=\frac{1}{a^{-n}}$$

$$a^n = 1 \div \frac{1}{a^n}$$

(Any number is equal to 1 divided by its reciprocal.)

$$1 \div \frac{1}{a^n} = 1 \div a^{-n}$$

(A negative exponent shows how many times the base is used as a divisor.)

$$\therefore \frac{a^n}{1}=\frac{1}{a^{-n}}$$

Hence, if a^{+n} is transferred to the denominator, it becomes a^{-n}; if a^{-n} is transferred to the numerator, it becomes a^{+n}.

Hence, the principle, Q.E.D.

1. Write with positive exponents: a^{-3}; x^{-4}; y^{-6}; a^2b^{-3}; $c^{-2}d$; a^3b^{-4}; $x^{-2}y^3$; x^3y^{-2}.

Ans. $\frac{1}{a^3}$; $\frac{1}{x^4}$; $\frac{1}{y^6}$; $\frac{a^2}{b^3}$; ....

2. Write the value of: a^0; $a^0b^0c^0x$; $\frac{1}{x^0}$; $(a+b)^0$; $\frac{a+b}{(a-b)^0}$; $(x+y-z)^0$.

Ans. 1; x; 1; 1; ....

3. Write with positive exponents: $\frac{x^2y^{-3}}{z^{-3}}$; $\frac{x^{-2}y^3}{z^3}$; $\frac{1}{a^{-2}b^3}$; $a^{-2}b^3$; $\frac{a^{-4}}{b^{-3}}$; $\frac{a^3}{b^{-4}}$; $\frac{x^{-1}y^{-2}}{z^{-3}}$; $\frac{ms^{-a}}{nt^{-b}}$.

Ans. $\frac{x^2z^3}{y^3}$; $\frac{y^3}{x^2z^3}$; $\frac{a^2}{b^3}$; ....

FRACTIONAL EXPONENTS

Proposition LXVI. Theorem

The numerator of a fraction shows that the base is to be affected by it in the manner denoted by the use of the fraction; the denominator calls for a quantity that will produce the base when that quantity is affected by the denominator in the same manner.

The relation of numerator to denominator must be the same in whatever way a fraction is used because the relation is independent of the use. Therefore, the general relation will be known if the relation is found when the fraction is used as a multiplier.

Let us take $$a \times \frac{m}{n}.$$

By definition, $a \times \frac{m}{n}$ means that a is to be multiplied by m and that the result is to be divided by n. That is, the numerator shows that the base is to be affected by it in the manner denoted by the use of the fraction.

By definition, to divide by n is to find the quantity that will produce the base when that quantity is multiplied by n. That is, the denominator calls for the quantity that will produce the base when that quantity is affected by the denominator in the manner denoted by the use of the fraction.

Hence, the principle, Q.E.D.

4. In $24 \div \frac{3}{4}$, what is the signification of each term of the fraction? What does the expression mean?

Ans. The numerator shows that 24 is to be divided by 3; the denominator calls for the number that will produce 24 when divided by 4. The expression means that 24 is to be divided by 3 and that a number is to be found which will produce that result when divided by 4. Or, it means that a number is to be found that will produce 24 when divided by 4, and that the result is to be divided by 3.

PROPOSITION LXVII. THEOREM

The numerator of a fractional exponent denotes the power to which the base is to be raised; the denominator denotes the root to which the base is to be depressed.

Let us take $$a^{\frac{m}{n}}$$

The numerator, m, shows that a is to be raised to the mth power.

(The numerator of a fraction shows that the base is to be affected by it in the manner denoted by the use of the fraction; the fraction in this position denotes the power to which the base is to be raised.)

The denominator, n, calls for the quantity that will produce a when this quantity is raised to the nth power.

(The denominator calls for a quantity that will produce the base when that quantity is affected by the denominator in the manner indicated by the use of the fraction.)

By definition, the quantity that will produce a when it is raised to the nth power is the nth root of a.

Hence, the principle, Q.E.D.

5. In $8^{\frac{2}{3}}$, what is the signification of each term of the fraction? What is the meaning of the expression?

Ans. The numerator shows that 8 is to be raised to the 2d power; the denominator calls for the number that will produce 8 when raised to the 3d power, or it calls for the cube root of 8. The expression means that 8 is to be raised to the 2d power and that the result is to be depressed to the 3d root, or that 8 is to be depressed to the 3d root and that the result is to be raised to the 2d power.

6. What is the value of $27^{\frac{2}{3}}$? Which is easier, to find the cube root of 27 and to square the result, or to square 27 and to find the cube root of the result?

NOTE. When the base is a perfect power, it is easier to extract the root first. The result must be the same as when the power is found first.

FRACTIONAL EXPONENTS

Express with radical signs:

7. $a^{\frac{2}{5}}b^{\frac{3}{4}}c^{\frac{1}{4}}$

8. $x^{\frac{2}{5}}y^{\frac{3}{4}}z^{\frac{1}{2}}$

9. $a^{\frac{4}{3}}b^{\frac{3}{5}}c^{\frac{2}{7}}$

10. $m^{\frac{1}{2}}n^{-\frac{3}{4}}$

11. $a^{\frac{2}{5}}b^{\frac{1}{2}}c^{\frac{2}{5}}$

12. $x^{\frac{m}{n}}y^{\frac{n}{m}}z^{\frac{m-1}{n-1}}$

Express with fractional exponents:

13. $\sqrt[5]{a^2}\times\sqrt[6]{b^5}$

14. $\sqrt[3]{a^8}\times\sqrt[4]{b^7}$

15. $\sqrt{x^5}\times\sqrt[6]{x^{11}}$

16. $\sqrt[11]{x^2}\times\sqrt[10]{y^3}$

17. $\sqrt[n]{a^m}\times\sqrt[m]{b^n}$

18. $\sqrt[n-1]{a^m}\times\sqrt[m]{b^{n-1}}$

Find the value of:

19. $(16)^{\frac{3}{4}}$; $(-27)^{\frac{5}{3}}$

20. $(81)^{\frac{5}{4}}$; $(-128)^{\frac{4}{7}}$

21. $(-243)^{\frac{2}{5}}$; $(-64)^{\frac{5}{3}}$

22. $(9)^{\frac{5}{2}}$; $(-32)^{\frac{3}{5}}$

23. $(125)^{\frac{4}{3}}$; $(-216)^{\frac{2}{3}}$

24. $(-8)^{\frac{8}{3}}$; $(-1024)^{\frac{4}{5}}$

Write with positive exponents:

25. $\dfrac{2x^{-3}y^{-4}}{3a^2b^2}$; $\dfrac{a^4b^{-1}c^4}{x^2y^{-4}z^{-5}}$

26. $\dfrac{4ax^{-1}}{5b^{-1}y^5}$; $\dfrac{m^{-5}n^{-6}}{5x^{-4}y^{-3}}$

27. $\dfrac{3a^{-1}b^3}{10c^{-5}d^2}$; $\dfrac{4m^2n^4}{5r^{-1}s^{-1}}$

28. $\dfrac{m^4n^{-2}}{5r^3s^{-6}}$; $\dfrac{3a^2b^{-3}}{4c^2d^{-1}}$

Ex. 7. $\sqrt[5]{a^2}\times\sqrt[4]{b^3c}$ Ex. 13. $a^{\frac{2}{5}}b^{\frac{5}{6}}$

Ex. 19. $\sqrt[4]{16}=\pm 2$; $(\pm 2)^3=\pm 8$. Or, $16^3=4096$; $\sqrt[4]{4096}=\pm 8$

MULTIPLICATION AND DIVISION

PROPOSITION XV. THEOREM *

To multiply when the bases are the same, write the common base, and over it, the exponent of the multiplicand plus the exponent of the multiplier.

NEGATIVE EXPONENTS

To prove that $a^{-m} \times a^{-n} = a^{-m-n}$

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} \qquad \text{Prop. VII}$$

$$= \frac{1}{a^{m+n}} \qquad \text{Prop. XLIV}$$

$$= a^{-m-n} \qquad \text{Prop. LXV}$$

Hence, the principle, Q.E.D.

FRACTIONAL EXPONENTS

To prove that $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{mq+np}{nq}}$

$$a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \times a^{\frac{np}{nq}} \qquad \text{Prop. XLII}$$

$$= \left(a^{\frac{1}{nq}}\right)^{mq} \times \left(a^{\frac{1}{nq}}\right)^{np} \qquad \text{Prop. LXVII}$$

$$= \left(a^{\frac{1}{nq}}\right)^{mq+np} \qquad \text{See note}$$

$$= a^{\frac{mq+np}{nq}} \qquad \text{Prop. LXVII}$$

Hence, the principle, Q.E.D.

NOTE. mq and np are positive integers; the proposition has been proved for this case.

* For positive exponents, see p. 31.

MULTIPLICATION AND DIVISION

Proposition XVI. Theorem *

To multiply when the exponents are the same, write the product of the bases, and over it the common exponent.

Negative Exponents

To prove that $a^{-m} \times b^{-m} = (ab)^{-m}$

$$a^{-m} \times b^{-m} = \frac{1}{a^m} \times \frac{1}{b^m} \qquad \text{Prop. VII}$$

$$= \frac{1}{(ab)^m} \qquad \text{Prop. XLIV}$$

$$= (ab)^{-m} \qquad \text{Prop. VII}$$

Hence, the principle, Q.E.D.

Fractional Exponents

To prove that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$

Let $x = a^{\frac{m}{n}}$ and $y = b^{\frac{m}{n}}$

To nth power, $x^n = a^m$ and $y^n = b^m$ Prop. L

Multiplying, $(xy)^n = (ab)^m$ Pos. Ex.

To nth root, xy, or $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$ Prop. LII

Hence, the principle, Q.E.D.

Simplify:

29. $a^{-3} \times a^2$; $a \times a^{-1}$; $x^2 \times x^{-3}$; $abc \times a^{-2}b^2c$; $a^{-4} \times a^4x$.

30. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$; $a^{\frac{3}{4}} \times a^{-\frac{1}{2}}$; $a^2b^{-\frac{1}{2}} \times b^{\frac{1}{2}}$; $x^{\frac{1}{3}}y^{\frac{2}{3}} \times x^{\frac{2}{3}}y^{\frac{1}{3}}$; $x^{-2}y \times -x^{\frac{1}{3}}y^{-\frac{3}{4}}$; $-a^{-\frac{1}{2}}b^{\frac{1}{3}} \times a^{\frac{1}{2}}b^{-\frac{1}{3}}$.

* For positive exponents, see p. 33.

In multiplication and division, the pupil should inquire, Are the bases the same? the exponents the same? or neither bases nor exponents the same? His method of procedure should be determined by the answer.

31. To divide when the bases are the same, write the common base and over it the exponent of the dividend minus the exponent of the divisor. Prove for negative exponents.*

SUGGESTION. Follow the same general method as on p. 167.

32. Prove the principle in Ex. 31 for fractional exponents.*

SUGGESTION. Follow the same general method as on p. 167.

33. To divide when the exponents are the same, write the quotient of the bases and over it the common exponent. Prove for negative exponents.†

SUGGESTION. Follow the same general plan as on p. 168.

34. Prove the principle in Ex. 33 for fractional exponents.†

SUGGESTION. Follow the same general plan as on p. 168.

Simplify:

35. $a^{-1} \div a^2$; $x^{-1} \div x^{-3}$; $a^{-1}b^3c^2 \div a^{-2}b^2c$; $x \div a^4x$; $a^0b^0c^0 \div -a^2b^{-3}c^4$; $a^{-3}b \div a^{-6}$.

36. $a^{\frac{5}{6}} \div a^{\frac{1}{3}}$; $a^{\frac{1}{4}} \div a^{-\frac{1}{2}}$; $a^2b^0 \div b^{\frac{1}{2}}$; $xy \times x^{\frac{1}{3}} \div y^{\frac{2}{3}}$; $-x^{-\frac{5}{3}}y^{\frac{1}{4}} \div x^{\frac{1}{3}}y^{-\frac{3}{4}}$; $a^0b^0 \div - a^{-\frac{1}{2}}b^{-\frac{1}{3}}$.

37. $a^{-2} \times b^{-2}$; $8^{-2} \div 4^{-2}$; $(x^2 - y^2)^{-3} \div (x - y)^{-3}$; $(a - b)^{-\frac{1}{2}} \times (a + b)^{-\frac{1}{2}}$; $(x^3 - 3x^2y + 3xy^2 - y^3)^{\frac{1}{2}} \div (x^2 - 2xy + y^2)^{\frac{1}{2}}$.

Ex. 37. $(x^3 - 3x^2y + 3xy^2 - y^3)^{\frac{1}{2}} \div (x^2 - 2xy + y^2)^{\frac{1}{2}}$. This is an example in division when the exponents are the same. The quotient of the bases is $x - y$; the answer is $(x - y)^{\frac{1}{2}}$.

* For positive exponents, see p. 37. † See p. 39.

MULTIPLICATION AND DIVISION

Multiply:

38. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{-\frac{1}{3}} - b^{-\frac{1}{3}}$

39. $x^{-1} + x^{-\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{-1} - x^{-\frac{1}{2}}y^{\frac{1}{2}} + y$

40. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$

Ex. 38.

$$
\begin{array}{r}
a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \qquad\qquad \\
- b^{-\frac{1}{3}} + a^{-\frac{1}{3}} \qquad\qquad \\
\hline
- a^{\frac{2}{3}}b^{-\frac{1}{3}} - a^{\frac{1}{3}} - b^{\frac{1}{3}} \qquad\qquad \\
a^{\frac{1}{3}} + b^{\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}} \\
\hline
- a^{\frac{2}{3}}b^{-\frac{1}{3}} \qquad + \qquad a^{-\frac{1}{3}}b^{\frac{2}{3}}
\end{array}
$$

We arrange the terms according to the descending powers of a. In the multiplier, $-b^{-\frac{1}{3}}$ should precede $a^{-\frac{1}{3}}$ because 0, the exponent of a in the former, is greater than $-\frac{1}{3}$, the exponent in the latter.

Divide:

41. $a^{-\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}b^{-\frac{1}{3}}$ by $a^{-\frac{1}{3}} - b^{-\frac{1}{3}}$

42. $x^{-2} + x^{-1}y + y^2$ by $x^{-1} + x^{-\frac{1}{2}}y^{\frac{1}{2}} + y^1$

43. $x^{\frac{5}{2}} + x^{\frac{3}{2}}y - xy^{\frac{3}{2}} - y^{\frac{5}{2}}$ by $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$

Ex. 41.

$$
\begin{array}{l}
- b^{-\frac{1}{3}} + a^{-\frac{1}{3}}\,)\, - a^{\frac{2}{3}}b^{-\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}}\,(\,a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} \\
\qquad\qquad\quad \underline{- a^{\frac{2}{3}}b^{-\frac{1}{3}} + a^{\frac{1}{3}} \qquad\quad} \\
\qquad\qquad\qquad\qquad - a^{\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}} \\
\qquad\qquad\qquad\qquad \underline{- a^{\frac{1}{3}} + b^{\frac{1}{3}} \qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad - b^{\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}} \\
\qquad\qquad\qquad\qquad\qquad \underline{- b^{\frac{1}{3}} + a^{-\frac{1}{3}}b^{\frac{2}{3}}}
\end{array}
$$

We arrange the terms in both dividend and divisor according to the descending powers of a and follow the same law with each partial dividend.

A number which can be expressed as an integer or as a fraction is rational; an indicated root of a positive number that cannot be expressed as an integer or a fraction is a surd; an indicated even root of a negative number is imaginary; both surds and imaginaries are irrational.

Thus, 6, $\frac{2}{3}$, .5, $\sqrt{4}$, $\sqrt[3]{8}$, 5^2 are rational numbers; $\sqrt{2}$, $\sqrt[3]{4}$ are surds; $\sqrt{2}$ is a quadratic surd; $\sqrt{-4}$, $\sqrt[4]{-3}$ are imaginaries. The value of a surd can be found true to any decimal place but cannot be exactly expressed by a decimal.

The case of *exponents the same* gives an ambiguous result in the multiplication and division of surds.

Thus, in multiplication by the case of *exponents the same*, $\sqrt{5} \times \sqrt{5} = \sqrt{25} = +5$, or -5. Of these two values, $+5$ alone is correct because the square of the square root of a number must be the number itself. In division, by the case of *exponents the same*, $\sqrt{5} \div \sqrt{5} = \sqrt{1} = \pm 1$. Of these two values, $+1$ alone is correct because anything divided by itself is 1.

It is customary, however, to use this case with surds, but to retain the positive value only in the root. For imaginaries and a more complete discussion, see pp. 246–248.

Thus, $\sqrt{5} \times \sqrt{20} = \sqrt{100} = +10$; $\sqrt{100} \div \sqrt{4} = \sqrt{25} = +5$.

Declare the case in multiplication or division:

44. $\sqrt{5} \times \sqrt[3]{5}$

45. $\sqrt[3]{6} \times \sqrt[3]{2}$

46. $\sqrt{3} \times \sqrt[3]{2}$

47. $\sqrt{5} \times \sqrt{5}$

48. $\sqrt[3]{5} \div \sqrt[3]{5}$

49. $\sqrt[3]{6} \div \sqrt[3]{2}$

50. $\sqrt{3} \div \sqrt[3]{2}$

51. $\sqrt{5} \div \sqrt{5}$

Ex. 44. This is the case in multiplication when the bases are the same.

Ex. 51. This is the case in division where either bases or exponents are the same. See Prop. XIX, p. 36.

SURDS — TERMS SIMPLIFIED

A surd is not in its simplest form when there is a perfect power of the same degree under the radical. We separate into factors, one of which is a perfect power, and perform the indicated operations.

Simplify:

52. $\sqrt[9]{27}$; $\sqrt[4]{4}$
53. $\sqrt[10]{64}$; $\sqrt[10]{32}$
54. $\sqrt[8]{81}$; $\sqrt[6]{125}$
55. $\sqrt[4]{49}$; $\sqrt[6]{25}$
56. $\sqrt[6]{81}$; $\sqrt[8]{16}$
57. $\sqrt{18\,a^3b^5}$; $\sqrt{50\,x^2y^3}$
58. $\sqrt{48\,a^5b^6}$; $\sqrt{45\,x^7y^6}$
59. $\sqrt[3]{16\,a^4b^6}$; $\sqrt[3]{81\,x^3y^7}$
60. $\sqrt[6]{27\,a^3b^3}$; $\sqrt[8]{49\,a^4x^2}$
61. $\sqrt[3]{54\,a^7b^4}$; $\sqrt{18\,ax^2}$

Ex. 52. Factoring, $\sqrt[9]{27} = (3^3)^{\frac{1}{9}}$; performing the indicated operation (Prop. L), $(3^3)^{\frac{1}{9}} = 3^{\frac{1}{3}} = \sqrt[3]{3}$.

Ex. 57. Factoring, $\sqrt{18\,a^3b^5} = \sqrt{9\,a^2b^4 \times 2\,ab}$; performing the indicated operation (Prop. LII), $\sqrt{9\,a^2b^4 \times 2\,ab} = 3\,ab^2\sqrt{2\,ab}$.

Simplify:

62. $\sqrt{a^3 - 2\,a^2b + ab^2}$
63. $\sqrt{2\,a^2 + 4\,ab + 2\,b^2}$
64. $\sqrt{a^3b + 2\,a^2b^2 + ab^3}$
65. $\sqrt{4\,a^2 - 24\,ab + 36\,b^2}$
66. $\sqrt{4\,ax^2 - 8\,a^2x + 4\,a^3}$
67. $\sqrt[3]{a^4 - 3\,a^3b + 3\,a^2b^2 - ab^3}$
68. $\sqrt[3]{2\,a^3 + 6\,a^2b + 6\,ab^2 + 2\,b^3}$
69. $\sqrt[3]{a^4b + 3\,a^3b^2 + 3\,a^2b^3 + ab^4}$
70. $\sqrt[3]{2\,x^4 + 6\,x^3y + 6\,x^2y^2 + 2\,xy^3}$
71. $\sqrt[4]{16\,x^2 + 64\,xy + 64\,y^2}$

Ex. 69. $(\ ) = \sqrt[3]{(a^3 + 3\,a^2b + 3\,ab^2 + b^3) \times ab} = (a + b)\sqrt[3]{ab}$

A surd is not in its simplest form when there is a fraction under the radical. It is necessary to multiply both numerator and denominator by something that will make the denominator a perfect power of the same degree.

Simplify:

72. $\sqrt{\frac{1}{2}}$; $\sqrt[3]{\frac{1}{2}}$

73. $\sqrt{\frac{1}{3}}$; $\sqrt[3]{\frac{1}{3}}$

74. $\sqrt{\frac{3}{11}}$; $\sqrt[3]{\frac{3}{11}}$

75. $\sqrt{\frac{5}{12}}$; $\sqrt[3]{\frac{5}{12}}$

76. $\sqrt{\frac{20}{27}}$; $\sqrt[3]{\frac{20}{27}}$

77. $\sqrt{\frac{25}{18}}$; $\sqrt[3]{\frac{5}{4}}$

78. $\sqrt[5]{\frac{3}{16}}$; $\sqrt{\frac{3}{20}}$

79. $\sqrt[4]{\frac{2}{9}}$; $\sqrt[3]{\frac{125}{54}}$

80. $\frac{3a^2}{4b}\sqrt{\frac{32b^3}{45a^2}}$

81. $\frac{7b^3}{5a^2}\sqrt{\frac{25a^6}{7b^3}}$

82. $b\sqrt{\frac{3(a-b)^2}{5b^3}}$

83. $(a+b)\sqrt{\frac{a-b}{a+b}}$

84. $\frac{a}{2}\sqrt[4]{\frac{8a}{27a^4}}$

85. $(x-y)\sqrt{\frac{x+y}{x-y}}$

86. $\frac{ax}{a-x}\sqrt{\frac{a^2-x^2}{ax}}$

87. $\frac{x+y}{x-y}\sqrt{\frac{x-y}{x+y}}$

Ex. 72. $\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=\sqrt{\frac{1}{4}\times 2}=\frac{1}{2}\sqrt{2}$; $\sqrt[3]{\frac{1}{2}}=\sqrt[3]{\frac{4}{8}}=\sqrt[3]{\frac{1}{8}\times 4}=\frac{1}{2}\sqrt[3]{4}$.

Ex. 80. $\frac{3a^2}{4b}\sqrt{\frac{32b^3}{45a^2}}=\frac{3a^2}{4b}\sqrt{\frac{160b^3}{225a^2}}=\frac{3a^2}{4b}\sqrt{\frac{16b^2}{225a^2}\times 10b}$

$$=\frac{3a^2}{4b}\times\frac{4b}{15a}\times\sqrt{10b}=\frac{a}{5}\sqrt{10b}.$$

SURDS — ADDITION AND SUBTRACTION

Before surds can be united, they must be reduced to equivalent expressions with a common surd factor. An even root is considered '+.' See p. 171.

Simplify:

88. $\sqrt{20}-\sqrt{45}+\sqrt{180}$

89. $\sqrt{50}-\sqrt{32}+\sqrt{8}$

90. $ab\sqrt{98\,a^3b^3}+\sqrt{242\,a^5b^5}$

91. $x^2y\sqrt{18\,x^3y^5}-xy^2\sqrt{8\,x^5y^3}$

92. $\sqrt{12}+3\sqrt{75}-2\sqrt{27}$

93. $\sqrt[3]{40}-\frac{1}{2}\sqrt[3]{320}+\sqrt[3]{135}$

94. $\sqrt{27\,xy^2}+\sqrt{75\,x^3}-\sqrt{3\,x^3-18\,x^2y+27\,xy^2}$

95. $\sqrt{3\,(x-y)^2}-\sqrt{12\,x^2+24\,xy+12\,y^2}+\sqrt{12\,x^2-24\,xy+12\,y^2}$

96. $x\sqrt{\dfrac{x-y}{x+y}}+y\sqrt{\dfrac{x+y}{x-y}}-(3\,y^2-x^2)\sqrt{\dfrac{1}{x^2-y^2}}$

Ex. 88

$$\begin{aligned}\sqrt{20}&=2\sqrt{5}\\ -\sqrt{45}&=-3\sqrt{5}\\ \sqrt{180}&=\underline{6\sqrt{5}}\\ &5\sqrt{5}\end{aligned}$$

Ex. 96

$$x\sqrt{\frac{x-y}{x+y}}=\frac{x}{x+y}\sqrt{x^2-y^2}$$

$$y\sqrt{\frac{x+y}{x-y}}=\frac{y}{x-y}\sqrt{x^2-y^2}$$

- - - - - - - - - - - - - -

Arrange in order of magnitude:

97. $\sqrt{2},\ \sqrt[3]{3},\ \sqrt[4]{5}$

98. $\sqrt{\frac{1}{2}},\ \sqrt[3]{\frac{1}{3}},\ \sqrt[4]{\frac{1}{5}}$

99. $\sqrt{125},\ \sqrt[3]{1332}$

100. $\sqrt[3]{\frac{2}{3}},\ \sqrt[2]{\frac{3}{5}},\ \sqrt[4]{\frac{1}{2}}$

Ex. 97. The surds must be reduced to equivalent surds with the same exponent. $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $5^{\frac{1}{4}}$, become $2^{\frac{6}{12}}$, $3^{\frac{4}{12}}$, $5^{\frac{3}{12}}$; or $\sqrt[12]{2^6}$, $\sqrt[12]{3^4}$, $\sqrt[12]{5^3}$; or $\sqrt[12]{64}$, $\sqrt[12]{81}$, $\sqrt[12]{125}$; the order is $\sqrt[4]{5}$, $\sqrt[3]{3}$, $\sqrt{2}$.

SURDS — MULTIPLICATION AND DIVISION

When the bases are the same it is well to use fractional exponents.

Multiply:

101. $\sqrt{7}$ by $\sqrt[3]{7}$

102. $\sqrt{6a^3b}$ by $\sqrt[3]{6a^3b}$

103. $\sqrt{63}$ by $\sqrt{7}$

104. $\sqrt{60a^3b^3}$ by $\sqrt{15ab}$

105. $\sqrt[3]{125xy}$ by $\sqrt[3]{5x^2}$

Divide:

106. $\sqrt{7}$ by $\sqrt[3]{7}$

107. $\sqrt{6a^3b}$ by $\sqrt[3]{6a^3b}$

108. $\sqrt{63}$ by $\sqrt{7}$

109. $\sqrt{60a^3b^3}$ by $\sqrt{15ab}$

110. $\sqrt[3]{125xy}$ by $\sqrt[3]{5x^2}$

Ex. 101. The bases are the same; $7^{\frac{1}{2}} \times 7^{\frac{1}{3}} = 7^{\frac{5}{6}} = \sqrt[6]{7^5}$.

Ex. 103. The exponents are the same; $\sqrt{63} \times \sqrt{7} = \sqrt{441} = 21$.

When neither bases nor exponents are the same, it is always possible to reduce to the case of exponents the same. It may be necessary to change the exponents to equivalent fractions having a common denominator.

Multiply:

111. $\sqrt{2}$ by $\sqrt[3]{3}$

112. $\sqrt[3]{4}$ by $\sqrt[4]{2}$

113. $8\sqrt{3}$ by $2\sqrt[3]{5}$

114. $\sqrt[3]{20}$ by $\sqrt{2}$

115. $\sqrt{2a}$ by $\sqrt[3]{3a^2}$

Divide:

116. $\sqrt{2}$ by $\sqrt[3]{3}$

117. $\sqrt[3]{4}$ by $\sqrt[4]{2}$

118. $8\sqrt{3}$ by $2\sqrt[3]{5}$

119. $\sqrt[3]{20}$ by $\sqrt{2}$

120. $\sqrt{2a}$ by $\sqrt[3]{3a^2}$

Ex. 120. Neither bases nor exponents are the same; $\sqrt{2a} \div \sqrt[3]{3a^2} = (2a)^{\frac{1}{2}} \div (3a^2)^{\frac{1}{3}} = (2a)^{\frac{3}{6}} \div (3a^2)^{\frac{2}{6}} = [(2a)^3]^{\frac{1}{6}} \div [(3a^2)^2]^{\frac{1}{6}} = (8a^3)^{\frac{1}{6}} \div (9a^4)^{\frac{1}{6}} = \sqrt[6]{\frac{8}{9a}}$

$$= \sqrt[6]{\frac{8 \times 3^4a^5}{9a \times 3^4a^5}} = \sqrt[6]{\frac{1}{3^6a^6} \times 648a^5} = \frac{1}{3a}\sqrt[6]{648a^5}.$$

SURDS — MULTIPLICATION AND DIVISION

It is sometimes necessary to place a factor within a radical.

Place factor within radical:

121. $-2\sqrt{3};\ -11\sqrt{2}$ **125.** $3a\sqrt[3]{3a^2};\ ac\sqrt[3]{a+c}$

122. $5\sqrt[3]{6x};\ 4\sqrt{3x}$ **126.** $a^2\sqrt[4]{ab};\ (x+y)\sqrt{3x}$

123. $6a\sqrt[3]{4a^4};\ 2\sqrt[4]{9}$ **127.** $5xy\sqrt{6xy};\ ab^n\sqrt{a^mb^m}$

124. $\dfrac{2a}{3b}\sqrt{\dfrac{18b^2}{8a^3}};\ \dfrac{a}{2}\sqrt[3]{\dfrac{16}{a^2}}$ **128.** $\dfrac{x}{x+y}\sqrt{x^2-y^2};\ \dfrac{x}{x-y}\sqrt{x^2-y^2}$

Ex. 121. This is an example in multiplication when neither bases nor exponents are the same, because $-2\sqrt{3}=-2^1\times 3^{\frac{1}{2}}$; making the exponents the same, $-2\sqrt{3}=-\sqrt{4}\times\sqrt{3}=-\sqrt{12}$. See p. 248.

It is sometimes necessary to multiply and to divide such expressions as $\sqrt{a}+\sqrt{b}$ and $\sqrt{c}+\sqrt{d}$. Multiplication is performed directly, but each partial product is simplified; division is preceded by multiplying both dividend and divisor by something that will free the divisor of surds. See p. 177 and p. 250.

Multiply:

129. $\sqrt{6}+\sqrt{24}$ by $\sqrt{12}-\sqrt{3}$

130. $3\sqrt{5}+2\sqrt{3}$ by $3\sqrt{5}-2\sqrt{3}$

131. $4\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-4\sqrt{b}$

132. $a\sqrt[3]{b^2}-d\sqrt[3]{d^2}$ by $a\sqrt[3]{b^2}+d\sqrt[3]{d^2}$

133. $\sqrt{5}-\sqrt{3}-\sqrt{2}$ by $\sqrt{5}+\sqrt{3}-\sqrt{2}$

Ex. 129.

$$\begin{array}{r}\sqrt{6}+\sqrt{24}\\ \sqrt{12}-\sqrt{3}\\ \hline 6\sqrt{2}+12\sqrt{2}\\ -\ 3\sqrt{2}-6\sqrt{2}\\ \hline 9\sqrt{2}\end{array}$$

$\sqrt{12}$ times $\sqrt{6}=\sqrt{72}=6\sqrt{2}$;
$\sqrt{12}$ times $\sqrt{24}=12\sqrt{2}$;
$-\sqrt{3}$ times $\sqrt{6}=-3\sqrt{2}$;
$-\sqrt{3}$ times $\sqrt{24}=-6\sqrt{2}$.

An expression is not in its simplest form when there is a surd in the denominator. It is necessary to multiply both numerator and denominator by something that will make the denominator rational.

To make the product rational, by what must we multiply:

134. $\sqrt{2}$? $\sqrt[3]{2}$?

135. $\sqrt[3]{3}$? $\sqrt[4]{5}$?

136. $\sqrt{18a^3}$? $\sqrt[3]{16a^4}$?

137. $\sqrt{5}+\sqrt{3}$? $\sqrt{5}-\sqrt{3}$?

138. $\sqrt{x}+\sqrt{y}$? $\sqrt{x}-\sqrt{y}$?

139. $x-\sqrt{y}$? $\sqrt{x}-\sqrt{y}$?

Ex. 134. The $\sqrt{2}$ must be multiplied by $\sqrt{2}$; $\sqrt{2}\times\sqrt{2}=2$.

Ex. 137. The $\sqrt{5}+\sqrt{3}$ must be multiplied by $\sqrt{5}-\sqrt{3}$; the product is $5-3$, or 2. The process exemplifies the product of the sum and difference of two quantities.

Simplify:

140. $\dfrac{1}{\sqrt{2}}$; $\dfrac{1}{\sqrt[3]{2}}$

141. $\dfrac{2}{\sqrt{5}}$; $\dfrac{2}{\sqrt[3]{5}}$

142. $\dfrac{6a^2}{\sqrt{18a^3}}$; $\dfrac{9a^2}{\sqrt{18a^5}}$

143. $\dfrac{15}{\sqrt{50x^3}}$; $\dfrac{10}{\sqrt[3]{25m^2}}$

144. $\dfrac{5}{\sqrt[3]{9c^2}}$; $\dfrac{2c}{\sqrt[4]{8c^3}}$

145. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{10}+\sqrt{6}}$

146. $\dfrac{\sqrt{8}-\sqrt{6}}{\sqrt{12}-\sqrt{3}}$

147. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

148. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

149. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Ex. 140. $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\dfrac{\sqrt{2}}{2}=\dfrac{1}{2}\sqrt{2}$; $\dfrac{1}{\sqrt[3]{2}}=\dfrac{\sqrt[3]{4}}{\sqrt[3]{2}\times\sqrt[3]{4}}=\dfrac{\sqrt[3]{4}}{\sqrt[3]{8}}=\dfrac{\sqrt[3]{4}}{2}=\dfrac{1}{2}\sqrt[3]{4}$.

Ex. 145. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{10}+\sqrt{6}}=\dfrac{(\sqrt{5}+\sqrt{3})(\sqrt{10}-\sqrt{6})}{(\sqrt{10}+\sqrt{6})(\sqrt{10}-\sqrt{6})}=\dfrac{2\sqrt{2}}{4}=\dfrac{1}{2}\sqrt{2}$.

SURDS — MULTIPLICATION AND DIVISION

A numerical expression, unless imaginary, is not in its simplest form when it involves any exponent other than 1.

Thus, 5^2 reduced to its simplest form is 25^1; $\sqrt[3]{8}$, 2^1; $\sqrt{2}$, 1.4142^1+

An expression which is to be simplified should be reduced as far as possible before any root is extracted.

Thus, $\sqrt[9]{27}=\sqrt[3]{3}$; it is easier to find the $\sqrt[3]{3}$ than the $\sqrt[9]{27}$.

$\frac{\sqrt{2}}{\sqrt{3}}=\frac{1}{3}\sqrt{6}$; it is easier to find the $\sqrt{6}$ and to divide the result by 3 than to find the $\sqrt{2}$ and the $\sqrt{3}$ and to divide.

$\frac{2}{\sqrt{3}}=\frac{2}{3}\sqrt{3}$; it is easier to find the $\sqrt{3}$ and to take $\frac{2}{3}$ of the result than to find the $\sqrt{3}$ and to divide 2 by the result.

Reduce to its simplest form (carry to 3 places):

150. $\sqrt{2}$; $\sqrt[3]{3}$

151. $\sqrt{50}$; $\sqrt[3]{24}$

152. $\sqrt{\frac{1}{2}}$; $\sqrt[3]{\frac{1}{9}}$

153. $\frac{3}{\sqrt{2}}$; $\frac{2}{\sqrt{3}}$

154. $\frac{\sqrt{2}-1}{\sqrt{2}+1}$; $\frac{\sqrt{2}+1}{\sqrt{2}-1}$

155. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$; $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

156. $\sqrt{5}$; $\sqrt[3]{5}$

157. $\sqrt{45}$; $\sqrt[3]{40}$

158. $\sqrt{\frac{1}{5}}$; $\sqrt{\frac{1}{5}}$

159. $\frac{2}{\sqrt{5}}$; $\frac{2}{\sqrt[3]{5}}$

160. $\frac{5-\sqrt{2}}{5+\sqrt{2}}$; $\frac{\sqrt{2}}{5+\sqrt{2}}$

161. $\frac{2\sqrt{3}-3\sqrt{2}}{5\sqrt{3}+4\sqrt{2}}$

Ex. 151. $\sqrt{50}=5\sqrt{2}=5\times 1.414+=7.070+$.

Ex. 155. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{5+2\sqrt{6}}{1}=5+2\times 2.449+=9.898+$.

INVOLUTION AND EVOLUTION

PROPOSITION L. THEOREM *

To raise a factor to any power, write the base, and over it the product of the exponent by the number denoting the required power.

NEGATIVE EXPONENTS

To prove that $(a^{-m})^{-n} = a^{mn}$

$$(a^{-m})^{-n} = \frac{1}{(a^{-m})^n} \qquad \text{Prop. VII}$$

$$= \frac{1}{a^{-mn}} \qquad \text{Positive Ex.}$$

$$= a^{mn} \qquad \text{Prop. LXV}$$

Hence, the principle, Q.E.D.

FRACTIONAL EXPONENTS

To prove that $(a^m)^{\frac{p}{q}} = a^{\frac{mp}{q}}$

$(a^m)^{\frac{p}{q}}$ means that a^m is to be raised to the p power and that the result is to be depressed to the q root. Prop. LXVII

$$(a^m)^p = a^{mp} \qquad \text{Positive Ex.}$$

$$\sqrt[q]{a^{mp}} = a^{\frac{mp}{q}} \qquad \text{Prop. LXVII}$$

Hence, the principle, Q.E.D.

Simplify:

162. $(a^{-2})^{-3}$; $(a^{-3})^{-2}$; $(ax^2)^{-3}$; $(x^3y^3)^{-2}$; $(x^{-2}y^{-3})^{-2}$; $(ab^{-1}c^{-2}d^2)^{-3}$; $[(a-b)^2]^{-1}$

163. $(a^{\frac{1}{3}})^{\frac{1}{2}}$; $(a^{-\frac{2}{3}})^{-\frac{1}{2}}$; $(a^2b^{-3})^{-\frac{1}{6}}$; $[(a^{\frac{1}{2}})^{-\frac{1}{3}}]^{-\frac{1}{5}}$; $[(a-b)^2]^{\frac{1}{2}}$; $([(a^{-2})^{-3}]^4)^{\frac{1}{24}}$

* For positive exponents, see p. 130.

SURDS — INVOLUTION AND EVOLUTION

The square of a binomial must consist in general of three terms, but may be reduced to two terms or even to one term. Thus,

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(2+\sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}.$$

$$(2+3)^2 = 4 + 12 + 9 = 25$$

It is necessary frequently to extract the square root of a binomial of the form $a+\sqrt{b}$ in which $\sqrt{b}$ is a surd. The obvious method is to find two expressions whose sum is a and twice the product of whose square roots is $\sqrt{b}$, for then $a+\sqrt{b}$ will become a trinomial which is a perfect square (p. 48). This may be done by inspection.

Find the square root of:

164. $7+4\sqrt{3}$	**167.** $12-6\sqrt{3}$	**170.** $1+4\sqrt{-3}$
165. $11-6\sqrt{2}$	**168.** $16+2\sqrt{55}$	**171.** $7-6\sqrt{-2}$
166. $7+2\sqrt{10}$	**169.** $5-2\sqrt{6}$	**172.** $49+12\sqrt{5}$

Ex. 164. $\quad 4\sqrt{3} = 2\sqrt{12}$

$$4,\ 3$$

$$2+\sqrt{3}$$

Since $4\sqrt{3}$ must be 2 times the product of the square roots of the parts, we will find an equivalent expression in which 2 shall be the only factor outside of the radical; what is within the radical will then be the product of the parts, and 7 will be their sum; $4\sqrt{3} = 2\sqrt{12}$. Hence, we must find two numbers whose product is 12 and whose sum is 7; they are 4 and 3. Then, $7 + 4\sqrt{3} = 4 + 4\sqrt{3} + 3$; the square root of 4 is 2; the square root of 3 is $\sqrt{3}$; twice their product is $4\sqrt{3}$; the square root of the expression is $2+\sqrt{3}$. See Prop. XXIII, p. 48.

NOTE. These examples may also be solved by Prop. LXVIII, as illustrated in Ex. 173.

SURDS — AN IMPORTANT PROPERTY

PROPOSITION LXVIII. THEOREM

If the sum of a rational term and a surd is equal to the sum of a rational term and a surd, the rational terms are equal and the surds are equal.

Let $x+\sqrt[n]{a}=y+\sqrt[n]{b}$ in which x and y are rational, and $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are surds. Transposing,

$$x-y=\sqrt[n]{b}-\sqrt[n]{a}$$

The left-hand member must be 0 or a rational quantity; the right-hand member must be 0 or an irrational quantity; the only common value is 0; $\therefore\ x-y=0$, and $\sqrt[n]{b}-\sqrt[n]{a}=0$; or $x=y$, and $\sqrt[n]{b}=\sqrt[n]{a}$.

Hence, the principle, Q.E.D.

This property affords another method of extracting the square root of $a+\sqrt{b}$.

173. Solve Ex. 164.

Let $$\sqrt{7+4\sqrt{3}}=\sqrt{x}+\sqrt{y}$$

Squaring, $$7+4\sqrt{3}=x+2\sqrt{xy}+y$$

Hence, $$x+y=7 \qquad (1)$$

and $$2\sqrt{xy}=4\sqrt{3} \qquad (2)$$

Equations (1) and (2) may be solved by the methods of simultaneous equations of higher degrees. See p. 208.

This property often affords a simple solution of a difficult example.

174. Solve $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=\sqrt{7}+\sqrt{3}$

Squaring, $$2+2x^2+2\sqrt{1+x^2+x^4}=10+2\sqrt{21}$$

Hence, $$2x^2+2=10 \qquad (1)$$

and $$2\sqrt{1+x^2+x^4}=2\sqrt{21} \qquad (2)$$

The equations may then be solved as quadratics. See p. 185.

EQUATIONS OF THE SECOND DEGREE

ONE UNKNOWN QUANTITY

An equation containing one unknown quantity is a quadratic equation or an equation of the second degree if the highest exponent of the unknown quantity is 2, and if there is no other exponent of the unknown quantity except 1. Thus,

$$5x^2 - 3x = 2, \quad (a + b)x^2 + (a - b)x = a^4$$

The general form of an equation of the second degree is

$$ax^2 + bx = c$$

An equation is also regarded as quadratic, with reference to some expression, if it can be reduced to a form in which the unknown quantity is found in only two terms and in which the exponent of the unknown quantity in one term is twice the exponent of the unknown quantity in the other term.

State why each of the following equations is quadratic and with reference to what expression:

1. $5x^6 + 4x^3 = 9$; $\sqrt{x} + \sqrt[4]{x} = 2$; $x^{-1} + x^{-\frac{1}{2}} = 2$
2. $3(x + 5)^2 - 5(x + 5) = -2$; $ax^3 + bx^{\frac{3}{2}} = c$; $x^{2n} + x^n = 2$

Ex. 1. $5x^6 + 4x^3 = 9$, is quadratic because the unknown quantity is found in only two terms and because the exponent of the unknown quantity in one term is twice the exponent of the unknown quantity in the other. It is quadratic with reference to x^3; it may be written, $5(x^3)^2 + 4(x^3)^1 = 9$.

REDUCTION TO $ax^2 + bx = c$

To solve an equation of the second degree, the first step is to reduce it to the form, $ax^2 + bx = c$. To reduce to this form, it may be necessary to employ various devices.

1. It may be necessary to transpose and unite.

Reduce to the form $ax^2 + bx = c$:

3. $5x + 7 = 3x^2 - x - 2$
4. $3x^2 + 2x - 9 = 47$
5. $3x^2 + 7x - 40 = 5x^2 - 12x + 4$
6. $x^2 - ax + a = ax^2 + x - a^3$
7. $acx^2 - bcx + adx = bd$
8. $ax^2 + ax - bx - bx^2 = a - b$

Ex. 6. Transposing, $x^2 - ax^2 - ax - x = -a^3 - a$; uniting, $(1 - a)x^2 - (1 + a)x = -a^3 - a$.

2. Equations must be cleared of fractions.

Reduce to the form $ax^2 + bx = c$:

9. $\dfrac{3}{x-2} - \dfrac{2}{x+2} = 2$
10. $\dfrac{x^2 + 9x}{15} = \dfrac{3(x+2)}{5}$
11. $\dfrac{x+2}{2x-1} = 2x + 1$
12. $\dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = \dfrac{2x+13}{x+1}$
13. $\dfrac{1}{3-x} - \dfrac{4}{5} = \dfrac{1}{9-2x}$
14. $\dfrac{3x-1}{4x+7} = 1 - \dfrac{6}{x+7}$
15. $\dfrac{1}{1-y} = \dfrac{8}{3} - \dfrac{1}{1+y}$
16. $\dfrac{x-2}{x+2} + \dfrac{x+2}{x-2} = 4$
17. $\dfrac{y-6}{7} + \dfrac{6}{21} = \dfrac{3(y+6)}{6y}$
18. $x^2 + \dfrac{a^2x - b^2x}{ab} = 1$
19. $\dfrac{x-4a}{2a-b} + \dfrac{2a+b}{x} = 0$
20. $\dfrac{x+3}{x-3} = 6\dfrac{6}{7} + \dfrac{x-3}{x+3}$

Ex. 9. Clearing, $3(x+2) - 2(x-2) = 2(x^2 - 4)$; transposing and uniting, $2x^2 - x = 18$.

REDUCTION TO $ax^2 + bx = c$

3. Sometimes, both members must be multiplied by the same expression.

Reduce to the form $ax^2 + bx = c$:

21. $5x^{-1} - 2x = x^{-1} + 2$

22. $x^{-\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{3}{2}}$

23. $3x^{-1} + 2 = 5x$

24. $2x^{-\frac{1}{3}} + x^{\frac{2}{3}} = x^{\frac{5}{3}}$

25. $3x^{-1} + 6 = 4x$

26. $5x^{-\frac{2}{3}} - 7x^{\frac{4}{3}} = x^{\frac{1}{3}}$

Ex. 22. Multiplying both members by $x^{\frac{1}{2}}$, $x^0 + x = 4x^2$; $4x^2 - x = 1$.

Note. The same results may be obtained if the equations are written with positive exponents and cleared of fractions. Thus, in Ex. 21; $\frac{5}{x} - 2x = \frac{1}{x} + 2$; clearing, $5 - 2x^2 = 1 + 2x$, $2x^2 + 2x = 4$, $x^2 + x = 2$.

4. Several terms may be placed within a parenthesis and the result may be regarded as an unknown quantity.

Reduce to the form $ax^2 + bx = c$:

27. $x^2 + 6 + \sqrt{x^2 + 12} = 0$

28. $3x^2 + x + \sqrt{3x^2 + x - 5} = 17$

29. $2x^2 + \sqrt{2x^2 + 1} = 11$

30. $x - 3\sqrt{2x - 1} = 2$

31. $x^2 + \sqrt{x^2 + 9} = 21$

32. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$

33. $x^2 - 2\sqrt{3x^2 - 2} = -1$

34. $x^2 + 2\sqrt{x^2 + x - 1} = -x$

35. $x^2 - \sqrt{3x^2 - 3x - 2} = x$

36. $4x^2 + 6\sqrt{4x^2 + 12x - 2} = -3 - 12x$

Ex. 28. We regard the part under the radical, $3x^2 + x - 5$, as the unknown quantity; subtracting 5 from each member, $3x^2 + x - 5 + \sqrt{3x^2 + x - 5} = 12$; placing in quadratic form, $(3x^2 + x - 5)^1 + (3x^2 + x - 5)^{\frac{1}{2}} = 12$.

The unknown quantity is $(3x^2 + x - 5)$; its exponent in the first term, 1, is twice its exponent in the second term, $\frac{1}{2}$; the unknown quantity is found in only two terms.

5. It may be necessary to square both members of the equation, or, as a preparation, to clear of fractions.

Reduce to the form $ax^2 + bx = c$*:*

37. $\sqrt{3x-1}+\sqrt{2x+6}=1$

38. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}$

39. $\dfrac{\sqrt{4x}+2}{4+\sqrt{x}}=\dfrac{4-\sqrt{x}}{\sqrt{x}}$

40. $\dfrac{\sqrt{x^2+9}+\sqrt{3x-3}}{\sqrt{x^2+9}-\sqrt{3x-3}}=4$

41. $\sqrt{2x+3}+\sqrt{3x+4}=2$

42. $\sqrt{9-2x}=\sqrt{x}-\sqrt{5-x}$

43. $\dfrac{\sqrt{2}-\sqrt{x}}{\sqrt{3}+\sqrt{x}}=\dfrac{\sqrt{3}+\sqrt{x}}{\sqrt{2}+\sqrt{x}}$

44. $\dfrac{\sqrt{5}}{\sqrt{2x^2-1}}=\dfrac{\sqrt{7}}{\sqrt{x+1}}$

45. $\dfrac{\sqrt{3x+1}}{\sqrt{x}}-\dfrac{\sqrt{2x-1}}{\sqrt{x}}=1$

46. $\dfrac{\sqrt{5x-1}}{\sqrt{x+2}}+\dfrac{\sqrt{x+2}}{\sqrt{5x-1}}=\dfrac{13}{6}$

Ex. 38. Squaring both members, $x+1+2\sqrt{x^2+17x+16}+x+16=x+25$; transposing and uniting, $2\sqrt{x^2+17x+16}=-x+8$; squaring again, $4(x^2+17x+16)=x^2-16x+64$; simplifying, $3x^2+84x=0$.

Ex. 39. Clearing of fractions, $2x+2\sqrt{x}=16-x$.

6. Sometimes when all the terms are transposed to the left-hand member, this member lacks a known quantity of being a perfect square.

Reduce to the form $ax^2 + bx = c$*:*

47. $x^4-4x^3-10x^2+28x=15$

48. $x^4+14x^3+47x^2-14x=48$

Ex. 47. By transposing and extracting the square root, we find that 64 is needed to complete the square; adding 64 to both members, $x^4-4x^3-10x^2+28x+49=64$, or $(x^2-2x-7)^2=64$.

COMPLETING THE SQUARE—FIRST METHOD

After the equation has been reduced to the form $ax^2 + bx = c$, it is necessary to obtain therefrom two equations of the form $ax = b$. This may be done by completing the square of the left-hand member and extracting the square root, or by factoring.

PROPOSITION LXIX. THEOREM

To complete the square, multiply the equation by something that will make the first term a perfect square, and add to each member the square of the quotient found by dividing the second term by twice the square root of the first.

Let us take $ax^2 + bx = c.$

If a trinomial is a perfect square, as $a^2 + 2ab + b^2$, its first term is a perfect square. Hence, we multiply the equation by a.

$$a^2x^2 + abx = ac$$

Its third term is the square of the quotient obtained from dividing the second term by twice the square root of the first. Hence, we add to each member $(abx \div 2\sqrt{a^2x^2})^2$, or $\frac{b^2}{4}$.

$$a^2x^2 + abx + \frac{b^2}{4} = ac + \frac{b^2}{4}$$

Hence, the principle, Q.E.D.

PROPOSITION LXX. THEOREM

After the square root has been extracted, only one member requires both the plus and the minus sign.

Let $\pm mx = \pm n$ be the square root.

To prove that $mx = \pm n$ is the equivalent.

Arranging from $\pm mx = \pm n$ as many equations as possible,

$$mx = +n \quad (1) \qquad\qquad -mx = -n \quad (3)$$
$$mx = -n \quad (2) \qquad\qquad -mx = +n \quad (4)$$

(3) becomes (1), and (4) becomes (2), when multiplied by -1.

Hence, the principle, Q.E.D.

Find the value of x and prove:

49. $3x^2 = 12$

50. $5x^2 = 125$

51. $ax^2 = 4a^3$

52. $(a-b)x^2 = a^2 - b^2$

53. $(a+b)x^2 = (a+b)^2$

54. $(a-b)^3x^2 = a-b$

Ex. 49. Dividing both members by 3, $x^2 = 4$; extracting the square root, $x = \pm 2$. PROOF. If $x = +2$, $x^2 = 4$, $3x^2 = 12$; if $x = -2$, $x^2 = 4$, $3x^2 = 12$.

NOTE. The above equations are sometimes called pure quadratics because the coefficient of x' is 0. The left-hand members may be made perfect squares by multiplying or dividing the equations.

Find the value of x:

55. $x^2 - 3x = -2$

56. $x^2 + x = 2$

57. $x^2 - 5x = -6$

58. $x^2 - 3ax = -2a^2$

59. $x^2 + ax = 2a^2$

60. $x^2 - ax = c$

Ex. 60. Completing the square, $x^2 - ax + \frac{a^2}{4} = \frac{a^2 + 4c}{4}$; extracting the square root, $x - \frac{a}{2} = \pm \frac{1}{2}\sqrt{a^2 + 4c}$; simplifying, $x = \frac{1}{2}(a \pm \sqrt{a^2 + 4c})$. One value of x is $\frac{1}{2}(a + \sqrt{a^2 + 4c})$; the other, $\frac{1}{2}(a - \sqrt{a^2 + 4c})$.

Find the value of x:

61. $2x^2 + 5x = 3$

62. $3x^2 - x = 2$

63. $6x^2 + 5x = 6$

64. $2x^2 + ax = a^2$

65. $3x^2 - 5ax = -2a^2$

66. $ax^2 + bx = c$

Ex. 66. Completing the square, $a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2 + 4ac}{4}$; extracting the square root, $ax + \frac{b}{2} = \pm \frac{1}{2}\sqrt{b^2 + 4ac}$; simplifying, $x = \frac{1}{2a}(-b \pm \sqrt{b^2 + 4ac})$.

COMPLETING THE SQUARE — SECOND METHOD

The square may be completed in such a manner as to avoid the use of fractions.

PROPOSITION LXXI. THEOREM

To complete the square, multiply the equation by four times the coefficient of x^2, and add to each member the square of the coefficient of x in the original equation.

Let us take $$ax^2 + bx = c \qquad (1)$$

Then $$a^2x^2 + abx + \frac{b^2}{4} = ac + \frac{b^2}{4} \qquad \text{Prop. LXIX}$$

Clearing of fractions, we obtain

$$4\,a^2x^2 + 4\,abx + b^2 = 4\,ac + b^2 \qquad (2)$$

Comparing (2) with (1), we see that the original equation has been multiplied by 4 times the coefficient of x^2, and that the square of the coefficient of x has been added to each side.

Hence, the principle, Q.E.D.

By this method, solve:

67 Ex. 61

68. Ex. 62

69. Ex. 63

70. Ex. 64

71. Ex. 65

72. Ex. 66

73. $13\,x^2 - 17\,x = -4$

74. $21\,x^2 + 22\,x = -5$

75. $9\,x^2 + 6\,x = 19$

76. $5\,x^2 - 4\,x = 105$

77. $21\,x^2 - 2\,ax = 3\,a^2$

78. $15\,x^2 + ax = 16\,a^2$

Ex. 72. Completing the square, $4\,a^2x^2 + (\;) + b^2 = 4\,ac + b^2$; extracting the square root, $2\,ax + b = \pm\sqrt{b^2 + 4\,ac}$; simplifying, $x = \frac{1}{2\,a}(-b \pm \sqrt{b^2 + 4\,ac})$.

Ex. 73. Completing the square, $4 \times 13^2x^2 - (\;) + 289 = -208 + 289$. It is better to indicate the product of 13 by 4×13 than to perform the multiplication, because the labor of multiplying is saved and it is easier to extract the square root of 4×13^2 than of 676.

NOTE. It is customary to represent the second term by a ( ), since its sign is all that needs to be known.

SOLUTION BY FACTORING

Proposition LXXII. Theorem

If a factor of both members of an equation contains an unknown quantity, this factor is equal to 0.

Let $x-a$, $x-b$, $x-c$, and so on to n expressions, be factors of both members of the equation,

$$(x-a)(x-b)(x-c)\cdots=0$$

To prove that $x-a=0$; $x-b=0$; $x-c=0$; $\cdots$

This equation is satisfied for every value of x that will make the first member 0. If $x=a$, the first member becomes $0\times(x-b)\times(x-c)\cdots$, or 0. If $x=b$, the first member becomes $(x-a)\times 0\times(x-c)\cdots$, or 0. Hence, $x=a$, $x=b$, and so on. Transposing, $x-a=0$; $x-b=0$; and so on.

Hence, the principle, Q.E.D.

Proposition LXXIII. Theorem

An equation with one unknown quantity has as many roots, or values of the unknown quantity, as there are units in the degree of the equation.

$$(x-a)(x-b)(x-c)\cdots=0 \tag{1}$$

may be written in the form

$$x^n+a_1x^{n-1}+b_1x^{n-2}\cdots+p_1x+q_1=0 \tag{2}$$

because the expansion of (1) will produce x^n followed by terms containing the descending powers of x and a term without x.

Since there are n values of x in (1) there must be n values of x in (2).

Hence, the principle, Q.E.D.

SOLUTION BY FACTORING

By this method solve:

79. Ex. 55
80. Ex. 56
81. Ex. 57
82. $x^2 - 1 = 0$
83. $x^3 - 1 = 0$
84. $x^2 - 5x = -6$
85. $x^2 + x = 56$
86. $(x^2 - 36)(x^2 - 4) = 0$
87. $(x - 7)(x - 2) = 0$
88. $x^2 - 35x = -300$
89. $x^4 - 2 + x^{-4} = 0$
90. $x^3 - 7x^2 = -12x$
91. $x^3 - x + 4x^2 - 4 = 0$
92. $x^3 + x^2 - 4x - 4 = 0$
93. $(x^2+5x+6)(x^2-3x+2)=0$
94. $(x^2 - a^2)(x^2 - b^2) = 0$
95. $(x^3 + a^3)(x^2 - a^2) = 0$
96. $x^2 + ax + bx = -ab$
97. $x^2 + 2a^2 - 3ax = 0$
98. $x^3 + 6x^2 + 12x + 8 = 0$
99. $6(2x-3)^2 - 12(2x-3) = 0$
100. $(2x^2)^2 - 7(2x^2) + 12 = 0$

Ex. 79. $x^2 - 3x = -2$; **transposing,** $x^2 - 3x + 2 = 0$; **factoring,**
$(x-1)(x-2) = 0$; $x - 1 = 0$, $x - 2 = 0$; $x = 1$ **or** 2

Ex. 83. Factoring, $(x-1)(x^2 + x + 1) = 0$; $x - 1 = 0$, $x^2 + x + 1 = 0$; **simplifying** $x^2 + x + 1 = 0$ **by completing the square,** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$; **from** $x - 1 = 0$, $x = 1$.

Ex. 90. Transposing, $x^3 - 7x^2 + 12x = 0$; **factoring,** $x(x-3)(x-4) = 0$; $x = 0$, $x - 3 = 0$, $x - 4 = 0$; $x = 0$, 3, **or** 4.

Find the value of x:

1. $x^2+2x=8$
2. $x^2-2x=3$
3. $3x^2-6x=9$
4. $3x^2+2x=56$
5. $2x^2-19x=-44$
6. $(1-a)x^2-(1+a)x=-a^3-a$

Ex. 6. Completing the square by the second method,

$$4(1-a)^2x^2 - (\ ) + (1+a)^2 = 4a^4 - 4a^3 + 5a^2 - 2a + 1$$
$$2(1-a)x - (1+a) = \pm(2a^2 - a + 1)$$
$$2(1-a)x = 1 + a + 2a^2 - a + 1, \text{ or}$$
$$1 + a - 2a^2 + a - 1$$
$$x = \frac{1+a^2}{1-a}, \text{ or } a$$

MISCELLANEOUS

Find the value of x:

7. $acx^2 + (ad - bc)x = bd$
8. $(a - b)x^2 + (a - b)x = a - b$
9. $2x^2 - x = 18$
10. $x^2 = 18$
11. $4x^2 - x = 3$
12. $5x^2 - 31x = -30$
13. $8x^2 - 55x = -78$
14. $x^2 - 9x = -14$
15. $8y^2 = 2$
16. $x^2 = 12$
17. $6y^2 - 9y = 162$
18. $abx^2 + (a^2 - b^2)x = ab$
19. $x^2 - 4ax = -4a^2 + b^2$
20. $4x^2 - 7x = 36$

Find the value of x:

21. $2x^2 + 2x = 4$
22. $4x^2 - x = 1$
23. $5x^2 - 2x = 3$
24. $x^2 - x = 2$
25. $4x^2 - 6x = 3$
26. $7x^2 + x = 5$

Find the value of x:

27. $(x^2 + 12)^1 + (x^2 + 12)^{\frac{1}{2}} = 6$
28. $(3x^2 + x - 5)^1 + (3x^2 + x - 5)^{\frac{1}{2}} = 12$
29. $(2x^2 + 1)^1 + (2x^2 + 1)^{\frac{1}{2}} = 12$
30. $(2x - 1)^1 - 6(2x - 1)^{\frac{1}{2}} = 3$
31. $(x^2 + 9)^1 + (x^2 + 9)^{\frac{1}{2}} = 30$
32. $(x^2 - 7x + 18)^1 + (x^2 - 7x + 18)^{\frac{1}{2}} = 42$
33. $(3x^2 - 2)^1 - 6(3x^2 - 2)^{\frac{1}{2}} = -5$
34. $(x^2 + x - 1)^1 + 2(x^2 + x - 1)^{\frac{1}{2}} = -1$

MISCELLANEOUS

Find the value of x:

35. $(3x^2-3x-2)^1-3(3x^2-3x-2)^{\frac{1}{2}}=-2$

36. $(4x^2+12x-2)^1+6(4x^2+12x-2)^{\frac{1}{2}}=-5$

Find the value of x:

37. $x^2-24x=-40$

38. $3x^2+84x=0$

39. $3x+2\sqrt{x}=16$

40. $9x^2-75x=-156$

41. $x^2-38x=39$

42. $2x^2-9x=-4$

43. $4x^2-8x=-1$

44. $14x^2-5x=12$

45. $2x^2-x=1$

46. $41x^2-99x=-34$

47. $(x^2-2x-7)^2=64$

48. $(x^2+7x-1)^2=49$

Find the value of x:

49. $\sqrt{7x+14}=\sqrt{2x+6}+\sqrt{x+4}$

50. $\sqrt{3x+7}-\sqrt{2x+3}=\sqrt{x-2}$

51. $x^2+3-\sqrt{2x^2-3x+2}=\frac{3}{2}(x+1)$

52. $x^3-3x^{\frac{3}{2}}=40$

53. $x^{+1}-5x^{-1}=4$

54. $x^{\frac{1}{2}}-2x^{-\frac{1}{2}}=1$

55. $x^{-2}-3x^{-1}=-2$

56. $x^4-8x^3+10x^2+24x+5=0$

57. $\dfrac{6x^2+6x-1}{x+1}-\dfrac{3x^2-15x+1}{x-5}=\dfrac{x^2+5x+1}{x+5}+\dfrac{2x^2-2x+1}{x-1}$

Note. Examples 3 to 48 are the answers to examples of the same numbers, pages 183 to 185.

SOLUTION OF PROBLEMS

When the relation between the terms of a problem must be expressed by an equation of the second degree, the solution of the equation gives two roots, both of which do not necessarily satisfy the problem; *i.e.*, a value that satisfies the equation may or may not satisfy the problem.

When the roots are both positive, both usually satisfy the problem.

1. The square of the number denoting how many persons are present plus 6 is equal to 5 times the number of persons. How many persons are present?

Let $x =$ the number; $x^2 + 6 = 5x$; $x = 2$ or 3; 2 persons or 3 persons are present. PROOF. $2^2 + 6 = 5 \times 2$; $3^2 + 6 = 5 \times 3$.

When the unit is not expressed, both roots usually satisfy the problem.

2. A number increased by its square becomes 12. Find the number.

Let $x =$ the number; $x^2 + x = 12$; $x = 3$ or -4. Both values satisfy the problem; $3^2 + 3 = 12$; $(-4)^2 + (-4) = 12$.

A negative root is usually to be rejected.

3. The square of the number denoting how many persons are present minus 6 is equal to 5 times the number of persons. How many persons are present?

Let $x =$ the number; $x^2 - 6 = 5x$; $x = 6$ or -1. Both values satisfy the equation but the negative does not satisfy the problem.

All of the problems in this exercise should be solved by the use of one unknown quantity. If two or more unknown quantities are used, simultaneous equations of higher degrees result, the solution of which has not yet been discussed.

SOLUTION OF PROBLEMS

If the sum of two numbers is given, it is often convenient to represent one of them by half the sum plus x and the other by half the sum minus x.

4. Prove that half the sum of two numbers plus x and half their sum minus x may represent any two numbers.

5. Divide 20 into two parts such that the difference of their squares shall be 160. Let $10 + x =$ the greater and $10 - x =$ the less.

6. Divide 20 into two parts such that their product is equal to the difference of their squares.

7. Divide $2a$ into two parts such that their product is equal to the difference of their squares.

8. The quotient of two numbers is 13 and their product is 637. What are the numbers?

9. The quotient of two numbers is a and their product is b. What are the numbers?

10. A certain number is formed by the product of three consecutive numbers, and if it be divided by each of them in turn, the sum of the quotients is 47. Find the number.

11. If one side of a square field be increased by 15 rods, and an adjacent side by 10 rods, the area of the field will be doubled. What is the area of the field?

12. A field whose length is 18 rods and whose width is 12 rods, is surrounded by a road, wholly without the field, whose area is equal to that of the field. Find the width of the road. Draw the field.

13. A field whose length is 18 rods and whose width is 12 rods is surrounded by a road, wholly within the field, whose area is equal to $\frac{5}{8}$ of the field. Find the width of the road. Draw the field.

14. A merchant sold goods for \$39, thereby gaining as many per cent as the goods cost him dollars. What was the cost?

Let $x =$ cost in dollars, 30

$$\frac{x}{100} = \text{gain per cent}$$

$$\frac{x^2}{100} = \text{gain in dollars}$$

$$x + \frac{x^2}{100} = 39$$

$$x = 30, \ -130$$

15. A person sold a horse for \$72; the number of per cent in his loss was $\frac{1}{8}$ of the number of dollars that he paid for the horse. What was the cost of the horse?

16. A certain sum is put at simple interest at 6%; if the number which denotes the dollars in the capital is multiplied by the number which denotes the cents in 5 months' interest, the product is 25,000. What is the principal?

17. A and B gained in trade \$18; A received in principal and gain \$154; B's principal was \$40. What was A's capital at the beginning?

18. At 6% simple interest, the amount of a principal for as many days as there are dollars in the principal is \$315. Find the principal.

19. The amount of x dollars for 2 years at x% is \$6.72. What is the principal?

20. At 6% simple interest, a dollars amounts to as many dollars as there are days in the time. What is the amount?

21. Two men were engaged at different rates of wages; the first received \$24; and the second, who worked 6 days less time, received \$13.50; if the first had worked 6 days less, and the second, 6 days more, they would have received the same sum. How many days did each work?

SOLUTION OF PROBLEMS

22. A merchant bought a certain number of pieces of silk for \$ 900; had he bought 3 pieces more for the same money, he would have paid \$ 15 less for each piece. How many pieces did he buy?

23. The expenses of a party of men amount to \$ 5.04; if each man pays 30 cents plus as many cents as there are persons, the bill will be settled. How many are there in the party?

24. A man bought a flock of sheep for \$ 100; if he had bought 5 more for the same sum, they would have cost \$ 1 less per head. How many did he buy?

25. A man bought a flock of sheep for \$ 100; if he had bought 5 less for the same sum they would have cost \$ 1 more per head. How many did he buy?

26. A man paid \$ 42.40 for having two piles of wood cut; there were 28 cords in all, and the labor cost as many dimes per cord as there were cords in the pile. How many cords were there in each pile?

27. A man worked a certain number of days for \$ 60; if he had received 50 cents a day less, he would have had to work 10 days longer to earn the same sum. How many days did he work?

28. A and B together can do a piece of work in $14\frac{2}{5}$ days; and A alone can do it in 12 days less than B alone. How long will it take A alone to do the work?

29. A courier travels from A to B in 14 hours; a second courier starts at the same time from a place 10 miles beyond A, and arrives at B at the same time as the first courier; the second courier takes one half hour less time to travel twenty miles than does the first courier. Find the distance from A to B.

30. A man had \$ 1 in silver and copper coins; each copper coin was worth as many cents as there were silver coins, and each silver coin was worth as many cents as there were copper coins; there were in all 27 coins. How many of each were there?

EQUATIONS OF HIGHER DEGREES

ONE UNKNOWN QUANTITY

The solutions of the general equation of the first degree, $ax + b = 0$, of the second degree, $ax^2 + bx = c$, and of the second degree with reference to some expression, $ax^{2n} + bx^n = c$, have been discussed in the preceding chapters. There are also direct solutions of the general equation of the third degree, $ax^3 + bx^2 + cx = d$, and of the fourth degree, $ax^4 + bx^3 + cx^2 + dx = e$, but these solutions are too difficult for an elementary treatise.

Equations higher than the fourth degree can be solved by trial only. The obvious method is to substitute for the unknown quantity, zero, each positive integer, and each negative integer. If a root is found to lie between two successive integers or between 0 and ± 1, it is necessary to substitute the decimals which lie between the limits. By this method, all real roots may be discovered, because every real root must be a positive number, a negative number, or zero. Devices for locating roots and for simplifying the method of trial form a science by themselves and may be found in more advanced works on algebra.

1. Find the real roots of $x^3 - 6x^2 - x + 30 = 0$.

Positive Integers. If $x = 0$, the equation becomes $30 = 0$; if $x = 1$, $24 = 0$; if $x = 2$, $12 = 0$; if $x = 3$, $0 = 0$; one root is 3. If $x = 4$, $-6 = 0$; if $x = 5$, $0 = 0$; one root is 5. If $x = 6$, $24 = 0$; if $x = 7$, $72 = 0$; since the discrepancy is increasing no root is > 5.

Negative Integers. If $x = -1$, $24 = 0$; if $x = -2$, $0 = 0$; one root is -2. It is unnecessary to search farther because an equation of the third degree can have only three roots. $\therefore x = 3, 5,$ or -2.

SOLUTION — REAL ROOTS

2. Locate the real roots of $x^3+3x^2-13x-38=0$.

Positive Integers. If $x=0$, $-38=0$; if $x=1$, $-47=0$; if $x=2$, $-44=0$; if $x=3$, $-23=0$; if $x=4$, $22=0$. Since it is necessary to pass through 0 in going from -23 to $+22$, one root lies between 3 and 4. If $x=5$, $97=0$; if $x=6$, $208=0$; no root is >4 because the discrepancy is increasing.

Negative Integers. If $x=-1$, $-23=0$; if $x=-2$, $-8=0$; if $x=-3$, $1=0$. Since it is necessary to pass through 0 in going from -8 to $+1$, one root lies between -2 and -3. If $x=-4$, $-2=0$. Since it is necessary to pass through 0 in going from $+1$ to -2, another root lies between -3 and -4.

There can be only 3 roots because the equation is of the third degree; one is between 3 and 4; a second, between -2 and -3; a third, between -3 and -4.

3. Locate the real roots of $x^5-1=0$.

Positive Integers. If $x=0$, the equation becomes $-1=0$; if $x=1$, $0=0$; one root is 1. If $x>1$ by any amount integral or decimal, $x^5>1$ and the left-hand member cannot become 0. The only positive root is 1.

Negative Integers. If $x=-1$ or any negative number integral or decimal, x^5 will become negative and the left-hand member cannot become 0. There is no real negative root.

The only real root is 1.

The value of the left-hand member for each supposed root can be found by division. If $x=a$, the equation is exactly divisible by $x-a$ (Prop. LXXII); therefore, if the equation is divided by $x-a$, the remainder must be the value of the left-hand member when a is substituted for x.

4. By division, find the value of the left-hand member when 1 is substituted for x in the equation, $x^3-6x^2-x+30=0$.

When x^3-6x^2-x+30 is divided by $x-1$, the quotient is x^2-5x-6 and the remainder is 24. Hence, 24 is the value of the left-hand member when 1 is substituted for x.

The value of the left-hand member can be found more easily than by substitution by the employment of synthetic division, an abbreviated process whereby the coefficients only are used.

Proposition LXXIV. Theorem

When the divisor is of the form $x \pm a$, the coefficient of each term of the quotient may be found by multiplying the coefficient of the preceding term by the coefficient of the second term of the divisor with changed sign, and adding the result to the corresponding coefficient of the dividend.

We will divide $x^3 - 6x^2 - x + 30$ by $x - 1$.

$$\begin{array}{l} 1 - 1)1 - 6 - 1 - 30(1 - 5 - 6 \\ \quad\quad \underline{1 - 1} \\ \quad\quad\;\; - 5 - 1 \\ \quad\quad\;\; \underline{-5 + 5} \\ \quad\quad\quad\quad - 6 + 30 \\ \quad\quad\quad\quad \underline{-6 + \;\;6} \\ \quad\quad\quad\quad\quad\quad 24 \end{array}$$

Analyzing the process, we see that the second term of the divisor is multiplied by each term of the quotient in succession and that the result is subtracted. This is equivalent to multiplying each term of the quotient in succession by the second term of the divisor with changed sign and adding.

Hence, the principle, Q.E.D.

Note. In practice, the work is arranged as below.

$$\begin{array}{cccc} 1 & -6 & -1 & +30 \\ 1 & \underline{\;\;1} & \underline{-5} & \underline{-\;\;6} \\ & -5 & -6 & +24 \end{array}$$

We write the coefficients of the dividend in a horizontal line. The first coefficient of the quotient is 1, which we write under 1; we multiply this by 1, the coefficient of the second term of the divisor with changed sign, add the result to -6, and obtain -5, the coefficient of the second term of the quotient; and so proceed. The quotient is $x^2 - 5x - 6$ and the remainder is 24

SOLUTION — REAL ROOTS

By synthetic division, find the value of the left-hand member of $x^3 - 6x^2 - x + 30 = 0$:

5. If $x = 2$

6. If $x = 3$

7. If $x = 4$

8. If $x = 5$

9. If $x = 6$

10. If $x = 7$

11. If $x = -1$

12. If $x = -2$

Ex. 11.

1	-6	-1	$+30$
1	-1	$+7$	-6
	-7	$+6$	24

If $x = -1$, we must divide by $x + 1$, and the constant multiplier will be -1. $-1 \times 1 = -1$; -1 and -6 are -7; and so on. The left-hand member becomes 24.

By synthetic division, find the value of the left-hand member of:

13. $x^5 - 1 = 0$, if $x = 1$; if $x = 2$; if $x = -1$

14. $x^3 + 3x^2 - 13x - 38 = 0$, if $x = 1$; if $x = 2$; if $x = -3$

15. $x^4 - 10x^2 - 4x + 8 = 0$, if $x = 1$; if $x = 2$; if $x = -3$

Ex. 13.

1	0	0	0	0	-1
1	1	1	1	1	1
	1	1	1	1	0

The coefficient of every wanting term must be represented by 0. $x^5 - 1 = x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1$.

16. By trial through the use of synthetic division, locate the roots of $x^4 - 10x^2 - 4x + 8 = 0$.

The sign of the left-hand member changes between 3 and 4, between -1 and -2, and between -2 and -3. If $x = 0$, $8 = 0$; if $x = 1$, $-5 = 0$; the sign also changes between 0 and 1. The roots are between 0 and 1, 3 and 4, -1 and -2, -2 and -3.

17. It has been found in Ex. 16 that one root of $x^4 - 10x^2 - 4x + 8 = 0$ is between 3 and 4. Find the root true to one decimal place and explain how you would proceed to find additional places.

If $x = 3.1$, the equation becomes $-.6521 = 0$. We may discover this by substituting 3.1 for x, but it is easier to use the method of detached coefficients.

$$\begin{array}{rrrrr} 1 & 0 & -10 & -4 & +8 \\ 1 & \underline{3.1} & \underline{9.61} & \underline{-1.209} & \underline{-16.1479} \\ & 3.1 & -\ \ .39 & -5.209 & -\ \ 8.1479 \end{array}$$

If $x = 3.2$, $-2.342 = 0$; if $x = 3.3$, $4.492 = 0$. One root lies between 3.2 and 3.3, or $x = 3.2$ true to one decimal place.

If we were to find additional decimal places, we should try 3.21, 3.22, 3.23, 3.24; finding that the sign changes with the latter, 3.23 would be the value of x true to two decimal places. We would then try 3.231, 3.232, and so on.

18. Find one root of $x^4 - 6x^3 + 27x^2 - 54x + 32 = 0$.

Ans. $x = 1$. $(x^4 - 6x^3 + 27x^2 - 54x + 32) \div (x - 1) = x^3 - 5x^2 + 22x - 32$.

19. Find one root of $x^3 - 5x^2 + 22x - 32 = 0$.

Ans. $x = 2$. $(x^3 - 5x^2 + 22x - 32) \div (x - 2) = x^2 - 3x + 16$.

20. Find the roots of $x^2 - 3x + 16$.

Ans. $x = \frac{1}{2}(3 \pm \sqrt{-55})$. This equation is solved as a quadratic.

21. What, then, are the four roots of $x^4 - 6x^3 + 27x^2 - 54x + 32 = 0$?

Ans. 1, 2, $\frac{1}{2}(3 + \sqrt{-55})$, $\frac{1}{2}(3 - \sqrt{-55})$.

NOTE. Reference is made to Exs. 18 to 21 on p. 204, Ex. 4.

22. Find one root of $x^4 - 22x^2 + x + 114 = 0$.

23. Divide $x^4 - 22x^2 + x + 114 = 0$ by $x - 3$ and find between what numbers the roots of the resulting equation lie.

Ans. $x^3 + 3x^2 - 13x - 38 = 0$. See Ex. 2.

NOTE. Reference is made to Exs. 22 and 23 on p. 210, Ex. 54.

SIMULTANEOUS EQUATIONS—HIGHER DEGREES

DISCUSSION

To solve simultaneous equations of higher degrees, it is necessary to find one equation with one unknown quantity. When there are only two equations with two unknown quantities, this may be done by various devices of which the more important are:

1. Finding the value of one of the unknown quantities in terms of the other in one equation, and substituting this value in the other equation.

2. Letting $y = vx$. This expedient is of value when the equations are homogeneous and of the second degree.

3. Letting $x = u + v$ and $y = u - v$. This expedient is of value when x and y are similarly involved in each equation.

4. Finding the values of $x + y$ and $x - y$.

An homogeneous equation is an equation in which all the terms except the absolute term are of the same degree.

Thus, $x^3 + y^3 = 9$ and $x^2y + xy^2 = 10$ are homogeneous equations; each term except the absolute (9 and 10) is of the third degree.

Symmetrical equations are equations in which x and y are similarly involved.

Thus, $x^3 + y^3 = 9$ and $x^2y + xy^2 = 10$ are symmetrical equations. $x^2 + xy - 3$ is not symmetrical. There is x^2 but no y^2.

The number of values of each unknown quantity is equal to the number of units in the degree of the equation which results from finding the value of one of the unknown quantities in terms of the other in one equation, and from substituting this value in the other equation.

How many values have x and y?

1. $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$ 2. $\begin{cases} x^3 + y^3 = 35 \\ x - y = 1 \end{cases}$

Ex. 1. $x = 5 - y$; substituting, $(5 - y)^3 + y^3 = 35$; since $-y^3$ and $+y^3$ cancel, this becomes an equation of the second degree; y will have two values and x will have two values.

Ex. 2. $x = 1 + y$; substituting, $(1 + y)^3 + y^3 = 35$; since y^3 and y^3 make $2y^3$, this becomes an equation of the third degree; y will have three values and x will have three values.

Proposition LXXV. Theorem

Any two unknown quantities may be represented by $u + v$ and $u - v$.

Let x and y be any two unknown quantities. Their sum may be represented by $2u$ because $2u$ may have every possible value; for the same reason, their difference may be represented by $2v$.

That is, $x + y = 2u$

and $x - y = 2v$

Adding, $2x = 2u + 2v$

Subtracting, $2y = 2u - 2v$

Dividing by 2, $x = u + v$

$y = u - v$

Hence, the principle, Q.E.D.

SOLUTION 1 — SUBSTITUTION

3. Solve by substitution: $\begin{cases} x^2 + y^2 = 13 & (1) \\ xy = 6 & (2) \end{cases}$

$$y = \frac{6}{x}$$

$$x^2 + \frac{36}{x^2} = 13$$

$$x^4 - 13x^2 = -36$$

$$x = \pm 3, \pm 2$$

$$y = \pm 2, \pm 3$$

From (2), $y = \frac{6}{x}\cdot$ Substituting in (1) and reducing, $x = \pm 3$, $y = \pm 2$.

NOTE 1. Each unknown quantity must have 4 values, because $x^4 - 13x^2 = -36$ is of the fourth degree.

NOTE 2. The values of the unknown quantities are to be taken in pairs, and should, therefore, be similarly arranged. Since $+3$ goes with $+2$, and -3 with -2, if $+3$ comes first in x, $+2$ must come first in y. It would not do to arrange them, $x = \pm 3$, $y = \mp 2$, for then $+3$ would go with -2; and, substituting in (2), -6 would equal $+6$, which is absurd.

4. Solve by substitution: $\begin{cases} x^4 + y^4 = 17 & (1) \\ x + y = 3 & (2) \end{cases}$

$$x^4 + (3 - x)^4 = 17$$

$$x^4 + 81 - 108x + 54x^2 - 12x^3 + x^4 = 17$$

$$2x^4 - 12x^3 + 54x^2 - 108x + 64 = 0$$

$$x^4 - 6x^3 + 27x^2 - 54x + 32 = 0 \quad (3)$$

$$x^3 - 5x^2 + 22x - 32 = 0 \quad (4)$$

$$x^2 - 3x + 16 = 0 \quad (5)$$

$$x = 1, 2, \tfrac{1}{2}(3 \pm \sqrt{-55})$$

By the methods of p. 201, Ex. 18, 1 is found to be a root of (3); dividing (3) by $x - 1$, we obtain (4). By the same methods, 2 is found to be a root of (4); dividing by $x - 2$, we obtain (5). Solving (5), we obtain the other two values of x, which are imaginary.

SOLUTION 2—LET $y = vx$

5. Solve the equations: $\begin{cases} x^2 + y^2 = 13 & (1) \\ \quad\;\; xy = \;\,6 & (2) \end{cases}$

Let

$$\begin{aligned} y &= vx \\ x^2 + v^2x^2 &= 13 \\ vx^2 &= \;\,6 \\ \frac{13}{1+v^2} &= \frac{6}{v} \qquad (3) \\ 6v^2 - 13v &= -6 \\ v &= \frac{2}{3}, \text{ or } \frac{3}{2} \end{aligned}$$

$$\begin{aligned} x^2 &= \frac{6}{v} = \frac{6}{\frac{2}{3}}, \text{ or } \frac{6}{\frac{3}{2}} \\ &= 9, \text{ or } 4 \\ x &= \pm 3, \text{ or } \pm 2 \\ y &= vx \\ &= \frac{2}{3}(\pm 3), \text{ or } \frac{3}{2}(\pm 2) \\ y &= \pm 2, \text{ or } \pm 3 \end{aligned}$$

Substituting vx for y, and equating the values of x^2 in the resulting equations, we obtain (3). Whence, $v = \frac{2}{3}$, or $\frac{3}{2}$.

Substituting these values for v in $x^2 = \frac{6}{v}$, $x = \pm 3$, or ± 2.

Since $y = vx$, $y = \frac{2}{3}$, the first value of v, times ± 3, the corresponding values of x; also $y = \frac{3}{2}$, the second value of v, times ± 2, the corresponding values of x.

Caution. Be sure to combine the value of v with the corresponding values of x. If $\frac{2}{3}$, the first value of v, is combined with ± 2, the second values of x, y would equal $\pm \frac{4}{3}$, values that will not satisfy the equations.

6. Solve the equations: $\begin{cases} 3x^2 + \;\;\; xy = \;\,5 & (1) \\ 4y^2 + 3xy = 22 & (2) \end{cases}$

Let

$$\begin{aligned} y &= vx \\ 3x^2 + vx^2 &= 5 \\ 4v^2x^2 + 3vx^2 &= 22 \\ \frac{5}{3+v} &= \frac{22}{4v^2 + 3v} \\ 5(4v^2 + 3v) &= 22(3 + v) \\ 20v^2 - 7v &= 66 \\ v &= 2, \text{ or } -\frac{33}{20} \end{aligned}$$

$$\begin{aligned} x^2 &= \frac{5}{3+v} = 1, \text{ or } \frac{100}{27} \\ x &= \pm 1, \text{ or } \pm\frac{10}{9}\sqrt{3} \\ y &= vx \\ &= 2 \times (\pm 1), \text{ or } -\frac{33}{20} \times \left(\pm\frac{10}{9}\sqrt{3}\right) \\ &= \pm 2, \text{ or } \mp\frac{11}{6}\sqrt{3} \end{aligned}$$

Note. This example should not be attempted by substitution.

SOLUTION 3—LET $x = u + v$ AND $y = u - v$

7\. Solve the equations: $\begin{cases} x^2 + y^2 = 13 & (1) \\ xy = 6 & (2) \end{cases}$

$$\begin{aligned} x^2 + y^2 &= 13 \quad (1) \\ xy &= 6 \quad (2) \end{aligned}$$

$$\begin{aligned} (u+v)^2 - (u-v)^2 &= 13 \\ (u+v)(u-v) &= 6 \end{aligned}$$

$$\begin{aligned} 2u^2 + 2v^2 &= 13 \\ u^2 - v^2 &= 6 \end{aligned}$$

$$\begin{aligned} 4u^2 &= 25 \quad (7) \\ 4v^2 &= 1 \quad (8) \end{aligned}$$

$$\begin{aligned} 2u &= \pm 5 \quad (9) \\ 2v &= \pm 1 \quad (10) \end{aligned}$$

$$\begin{aligned} 2u + 2v &= \pm 6, \text{ or } \pm 4 \\ 2u - 2v &= \pm 4, \text{ or } \pm 6 \end{aligned}$$

$$\begin{aligned} x &= \pm 3, \text{ or } \pm 2 \\ y &= \pm 2, \text{ or } \pm 3 \end{aligned}$$

Note 1. It is well to keep the equations in pairs. Thus, it is better to place $4v^2$ under $4u^2$ as in (7) and (8), than from (7) to find $u^2 = \frac{25}{4}$, $u = \pm\frac{5}{2}$, and then from (8), $v^2 = \frac{1}{4}$, $v = \pm\frac{1}{2}$. The above arrangement keeps the values together.

Note 2. In subtracting (10) from (9), care must be taken to use the terms in exactly the same order as in adding.

8\. Solve the equations: $\begin{cases} x^4 + y^4 = 17 & (1) \\ x + y = 3 & (2) \end{cases}$

$$\begin{aligned} (u+v)^4 + (u-v)^4 &= 17 \\ (u+v) + (u-v) &= 3 \end{aligned}$$

$$\begin{aligned} 2u^4 + 12u^2v^2 + 2v^4 &= 17 \\ 2u &= 3 \end{aligned}$$

$$u = \tfrac{3}{2}$$

$$2(\tfrac{3}{2})^4 + 12(\tfrac{3}{2})^2 v^2 + 2v^4 = 17$$

$$\tfrac{81}{8} + 27v^2 + 2v^4 = 17$$

$$v = \pm\tfrac{1}{2}, \text{ or } \pm\tfrac{1}{2}\sqrt{-55}$$

$$u = \tfrac{3}{2}$$

$$\begin{aligned} x &= 2,\ 1,\ \tfrac{1}{2}(3 \pm \sqrt{-55}) \\ y &= 1,\ 2,\ \tfrac{1}{2}(3 \mp \sqrt{-55}) \end{aligned}$$

Note. The pupil should compare this solution with that of p. 204. This device affords a solution without appeal to equations of higher degrees.

SOLUTION 4 — FINDING $x + y$ AND $x - y$

9. Solve the equations: $\begin{cases} x^2 + y^2 = 13 & (1) \\ xy = 6 & (2) \end{cases}$

$$\begin{aligned} 2xy &= 12 && (3) \\ x^2 + 2xy + y^2 &= 25 && (4) \\ x^2 - 2xy + y^2 &= 1 && (5) \\ x + y &= \pm 5 \\ x - y &= \pm 1 \end{aligned}$$

$$\begin{aligned} 2x &= \pm 6, \text{ or } \pm 4 \\ 2y &= \pm 4, \text{ or } \pm 6 \\ x &= \pm 3, \text{ or } \pm 2 \\ y &= \pm 2, \text{ or } \pm 3 \end{aligned}$$

By adding (3) to (1), we obtain (4); by subtracting (3) from (1), we obtain (5). We then extract the square root of each, and reduce.

NOTE. It thus appears that some examples may be solved in each of the four ways. The fourth is the simplest for this example.

10. Solve the equations: $\begin{cases} x^4 + y^4 = 17 & (1) \\ x + y = 3 & (2) \end{cases}$

$$\begin{aligned} x^4 + y^4 &= 17 \\ x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 &= 81 \\ 4x^3y + 6x^2y^2 + 4xy^3 &= 64 \\ 4x^3y + 8x^2y^2 + 4xy^3 &= 36xy && (3) \\ 2x^2y^2 - 36xy &= -64 \\ x^2y^2 - 18xy &= -32 && (4) \\ xy &= 2, \text{ or } 16 && (5) \\ x + y &= 3 \end{aligned}$$

$$\begin{aligned} x^2 + 2xy + y^2 &= 9 \\ x^2 - 2xy + y^2 &= 1, \text{ or } -55 \\ x + y &= 3 \\ x - y &= \pm 1, \text{ or } \pm\sqrt{-55} \\ 2x &= 4, \quad 2, \quad 3 \pm \sqrt{-55} \\ 2y &= 2, \quad 4, \quad 3 \mp \sqrt{-55} \\ x &= 2, \quad 1, \quad \tfrac{1}{2}(3 \pm \sqrt{-55}) \\ y &= 1, \quad 2, \quad \tfrac{1}{2}(3 \mp \sqrt{-55}) \end{aligned}$$

To obtain (3), we square (2) and multiply by $4xy$.
Solving (4) as a quadratic, we obtain (5).
Squaring (2) and subtracting 4 times (5), we obtain the square of $x - y$.
From the values of $x + y$ and $x - y$ the values of x and y are readily found.

NOTE. The pupil should make a comparison and careful study of these different solutions.

SOLUTION

The example should be studied carefully to discover which of the four devices will best apply. Nothing should be written until the full plan of solution has been formed.

State the plan for solving:

11. $\begin{cases} x^2 + 2xy - 3y^2 = 5 \\ 2x^2 + 5y^2 = 13 \end{cases}$

12. $\begin{cases} x^4 + x^2y^2 + y^4 = 133 \\ x^2 + xy + y^2 = 19 \end{cases}$

13. $\begin{cases} x^5 - y^5 = 31 \\ x - y = 1 \end{cases}$

14. $\begin{cases} x^3 + y^3 = 9 \\ x + y = 3 \end{cases}$

15. $\begin{cases} x^2 + y^2 = 5 \\ x + y = 3 \end{cases}$

16. $\begin{cases} 3x^2 + 2y^2 = 11 \\ 5x^2 + y^2 = 9 \end{cases}$

Ex. 11. *Solution 2.* The equations are homogeneous and of the second degree; let $y = vx$. No one of the other devices is applicable.

Ex. 12. *Solution 3.* The equations are symmetrical, *i.e.* x and y are similarly involved in each; let $x = u + v$ and $y = u - v$.

Solution 4. To find $x + y$ and $x - y$ is a better device; dividing (1) by (2), $x^2 - xy + y^2 = 7$; the addition and subtraction in turn of (2) and this result will give $2x^2 + 2y^2 = 26$ and $2xy = 12$.

If x^2y^2 is transposed in (1), and xy in (2), and the square of the last result is subtracted from (1) as transposed, a quadratic in xy will result. It will then be easy to find $x + y$ and $x - y$.

Ex. 13. *Solution 3.* The equations are symmetrical; let $x = u + v$ and $y = u - v$.

Solution 4. Since $x^5 - y^5$ is divisible by $x - y$, (1) may be divided by (2) to employ the device of finding $x + y$ and $x - y$. See Ex. 10.

Since the fifth power of $x - y$ has $-y^5$ for its last term, the subtraction of the fifth power of (2) from (1) will also permit the use of the fourth device. See Ex. 10.

Solution 2. The value of y in terms of x in (2) may be substituted in (1). See Ex. 4.

Ex. 16. Multiply (1) by 5 and (2) by 3, and subtract to eliminate x^2.

SOLUTION

Solve:

17. Ex. 11 by letting $y = vx$.

18. Ex. 12 by letting $x = u + v$ and $y = u - v$.

19. Ex. 12 by dividing (1) by (2), and by adding and subtracting the resulting equation and (2).

20. Ex. 12 by dividing (1) by (2), and letting $y = vx$ in the resulting equation and in (2).

21. Ex. 12 by transposing x^2y^2 to the right-hand member in (1), xy in (2), squaring the last result, and subtracting from (1) as transposed.

22. Ex. 13 by letting $x = u + v$ and $y = u - v$.

23. Ex. 13 by dividing (1) by (2).

24. Ex. 13 by raising (2) to the 5th power.

25. Ex. 13 by substituting in (1) the value of y in terms of x as found in (2).

26. Ex. 13 by substituting in (1) the value of x in terms of y as found in (2).

27. Ex. 14 by letting $x = u + v$ and $y = u - v$.

28. Ex. 14 by dividing (1) by (2).

29. Ex. 14 by raising (2) to the third power.

30. Ex. 15 by substituting in (1) the value of y in terms of x as found in (2).

31. Ex. 15 by letting $x = u + v$ and $y = u - v$.

32. Ex. 15 by raising (2) to the second power.

33. Ex. 16 by letting $y = vx$.

34. Ex. 16 by eliminating x^2 from (1) and (2).

SOLUTION — MISCELLANEOUS

Solve:

35. $\begin{cases} 3x + y = 9 \\ x^2 + y^2 = 13 \end{cases}$

36. $\begin{cases} x^2 - xy = 15 \\ 2xy - y^2 = 16 \end{cases}$

37. $\begin{cases} x + y = 5 \\ x^4 + y^4 = 97 \end{cases}$

38. $\begin{cases} x^3 - y^3 = 19 \\ x^2y - xy^2 = 6 \end{cases}$

39. $\begin{cases} x^2 + y^2 = 20 \\ x - y = 2 \end{cases}$

40. $\begin{cases} x + y = 12 \\ x^3 + y^3 = 18xy \end{cases}$

41. $\begin{cases} x + y = 6 \\ x^3 - y^3 = 7(x - y)^3 \end{cases}$

42. $\begin{cases} x - y = 4 \\ (x + y)^2 + (x + y) = 20 \end{cases}$

43. $\begin{cases} x^2 + y^2 - x - y = 14 \\ xy + x + y = 14 \end{cases}$

44. $\begin{cases} x^2 + xy + y^2 = 333 \\ x - y = 3 \end{cases}$

45. $\begin{cases} xy = 12 \\ x^2 + y^2 + 3\sqrt{x^2 + y^2} = 40 \end{cases}$

46. $\begin{cases} x^3 + y^3 = 9 \\ x^2 - xy + y^2 = 3 \end{cases}$

47. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 3 \\ \dfrac{1}{x^3} + \dfrac{1}{y^3} = 9 \end{cases}$

48. $\begin{cases} x^2 + xy - y^2 = 11 \\ 2x^2 - xy + y^2 = 16 \end{cases}$

49. $\begin{cases} x - y = 4 \\ x^2 + y^2 = 40 \end{cases}$

50. $\begin{cases} x^2 - y^2 = 8 \\ xy - y^2 = 2 \end{cases}$

51. $\begin{cases} 3x^2 - 2xy = 5 \\ x - y = 2 \end{cases}$

52. $\begin{cases} x + y = 5 \\ 2x^2 + y^2 = 17 \end{cases}$

53. $\begin{cases} x = 2y^2 \\ x - y = 15 \end{cases}$

54. $\begin{cases} x^2 + y = 11 \\ x + y^2 = 7 \end{cases}$

55. $\begin{cases} x + y + \sqrt{x + y} = 12 \\ x - y + \sqrt{x - y} = 2 \end{cases}$

56. $\begin{cases} x + y + z = 9 \\ x^2 + y^2 + z^2 = 29 \\ x^3 + y^3 + z^3 = 99 \end{cases}$

Ex. 54. Substituting in (2) the value of y as found in (1), $x^4 - 22x^2 + x + 114 = 0$. See p. 201, Ex. 22.

SOLUTION OF PROBLEMS

All of the problems in quadratics involving more than one condition may be solved by the use of two or more unknown quantities. To attempt the solution of the problems in this exercise by the use of a single letter would involve unnecessary labor.

1. The difference of the squares of two numbers is 5; the sum of their squares is 13. Find the numbers.

ONE

Let $x =$ the greater, ± 3

$\sqrt{13 - x^2} =$ the less, ± 2

$$x^2 - (13 - x^2) = 5$$
$$2x^2 = 18$$
$$x^2 = 9$$
$$x = \pm 3$$

TWO

Let $x =$ greater, $y =$ less

$$x^2 - y^2 = 5$$
$$x^2 + y^2 = 13$$
$$2x^2 = 18$$
$$2y^2 = 8$$
$$x = \pm 3,\ y = \pm 2$$

2. (*a*) The product of two numbers plus the square of the greater is 15; the product of the numbers plus the square of the less is 10. Let $x =$ the greater and find the value of the less in terms of x.

(*b*) Find the equation which must be solved to get the values by this method.

(*c*) Solve the equation and find the numbers.

(*d*) Let x equal the greater and y equal the less and solve the example.

3. The sum of two numbers is 7 and their product is 12. Find the numbers.

4. The difference of two numbers is 3 and their product is 10. Find the numbers.

5. The product of the sum and difference of two numbers is 21 and the square of their difference is 9. Find the numbers.

SOLUTION OF PROBLEMS

6. The difference of two numbers is 2 and the difference of their cubes is 56. Find the numbers.

7. The sum of two numbers is 5 and the sum of their cubes is 35. Find the numbers.

8. Extract the square root of $8 + 4\sqrt{3}$. See p. 181.

9. The product of two numbers is 48 and the difference of their cubes is 37 times the cube of their difference. What are the numbers?

10. A man sold a number of sheep for \$160; had he reserved 5 and sold the remainder at \$1 apiece more, he would have received \$135. How many did he sell and at what price?

11. A and B bought 600 acres of land for \$600, each paying \$300; in dividing, A took the best land and paid \$$\frac{3}{4}$ more an acre than B. How many acres did each get and at what price?

12. The distance around a rectangular field is 500 yards and its area is 14,400 square yards. Find the dimensions of the field.

13. The hypotenuse of a right-angled triangle is 25 feet and its area 150 square feet. Find its base and perpendicular.

14. A tree 80 feet high is broken so that the broken part rests on the stump and the top of the tree touches the ground 40 feet from the stump. Find the height of the stump if the ground is level.

15. There are three rectangles, in each of which the length is twice the width; the sum of their lengths is 48; the sum of their areas is 400; and the area of the third is equal to the sum of the areas of the first and second. Find the area of each.

16. The front wheel of a carriage makes 6 revolutions more than the hind wheel in going 360 feet; if the circumference of each wheel had been 3 feet greater the front wheel would have made only 4 revolutions more than the hind wheel in going that distance. What is the circumference of each wheel?

RATIO AND PROPORTION

DEVELOPMENT

That case in division in which both dividend and divisor are of the same denomination may be expressed by the symbol ':'. The expression is a ratio; the dividend, the antecedent; the divisor, the consequent.

$$6 \text{ ft.} : 2 \text{ ft.}$$

$$6 : 2$$

$$a : b$$

The above are ratios. The quotient always expresses times or activities. The first ratio is read 6 ft. is to 2 ft., and means that 6 feet contain 2 feet 3 times; the second means that 6 units of every kind contain 2 units of the same kind 3 times; the third means that every number of units of every kind contains every number of units of the same kind some number of times.

There may be a ratio of a ratio, a compound ratio.

$$\left.\begin{matrix} 6 \text{ ft.} : 2 \text{ ft.} \\ 5 \text{ men} : 3 \text{ men} \end{matrix}\right\} \qquad \left.\begin{matrix} 6 : 2 \\ 5 : 3 \end{matrix}\right\} \qquad \left.\begin{matrix} a : b \\ c : d \end{matrix}\right\}$$

These are compound ratios. We speak of multiplying the antecedents for a new antecedent and the consequents for a new consequent.

$$\frac{6 \text{ ft.} \times 5 \text{ men}}{2 \text{ ft.} \times 3 \text{ men}}, \text{ or } \frac{30 \text{ ft.} \times \text{men}}{6 \text{ ft.} \times \text{men}} = 5$$

Strictly speaking, this is impossible; feet cannot be multiplied by men. It is evident, however, that the numerator contains the denominator 5 times; no attempt is made to perform the multiplication of the denominate units, though the coefficients may be multiplied.

DEVELOPMENT

Two ratios may be equal, a proportion. The first and last terms are extremes; the second and third terms, means; the fourth term, a fourth proportional.

$$6 \text{ ft.} : 2 \text{ ft.} = 12 \text{ men} : 4 \text{ men}$$

$$a : b :: c : d$$

The first proportion means that 6 ft. contain 2 ft. the same number of times that 12 men contain 4 men. The symbol '::' is often used for '=' and read *as*. The second proportion is read a is to b as c is to d.

If the means of a proportion are the same, either mean is a mean proportional to the extremes; either extreme is a third proportional to the mean and the other extreme.

$$a : b :: b : c$$

b is a mean proportional to a and c; c is a third proportional to a and b.

In a series of equal ratios, each consequent may be the same as the following antecedent, a continued proportion.

$$a : b = b : c = c : d = d : e$$

Every proposition in regard to a proportion may be made of an equality of two fractions.

Thus, Prop. LXXVI. If two fractions are equal, the product of the numerator of the first by the denominator of the second is equal to the product of the numerator of the second by the denominator of the first. That is,

If $\frac{2}{3} = \frac{4}{6}$	If $\frac{a}{b} = \frac{c}{d}$
$2 \times 6 = 4 \times 3$	$a \times d = c \times b$

The numerators correspond to the antecedents; the denominators correspond to the consequents.

NOTE. — The pupil is requested to state each of the following propositions in regard to an equality of two fractions that he may see the advantage of the phraseology of proportion.

PROPORTION — PRINCIPLES

Proposition LXXVI. Theorem

In any proportion, the product of the extremes is equal to the product of the means.

Let $a:b::c:d$

To prove that $ad = bc$

$$\frac{a}{b} = \frac{c}{d}$$

(We write the proportion in fractional form.)

$\therefore$ $ad = bc$

(We clear of fractions.)

Hence, the principle, Q.E.D.

1. State as an equality of two fractions. See p. 214.

Proposition LXXVII. Theorem

If the product of two quantities is equal to the product of two others, one pair may be made the extremes and the other pair the means of a proportion.

Let $ad = bc$

To prove that $a:b::c:d$

$$\frac{ad}{bd} = \frac{bc}{bd}$$

(We divide the original equation by bd.)

$$\frac{a}{b} = \frac{c}{d}$$

(We simplify.)

$\therefore$ $a:b::c:d$

(We write in proportional form.)

Hence, the principle, Q.E.D.

2. State as an equality of two fractions. See p. 214.

PROPORTION — PRINCIPLES

PROPOSITION LXXVIII. THEOREM

A mean proportional between two quantities is equal to the square root of their product.

Let $a : b :: b : c$

To prove that $b = \sqrt{ac}$

$$b^2 = ac$$

(The product of the extremes equals the product of the means.)

$$\therefore \quad b = \sqrt{ac}$$

(We extract the square root of both members.)

Hence, the principle, Q.E.D.

3. State as an equality of two fractions. See p. 214.

PROPOSITION LXXIX. THEOREM

In any proportion, the terms are in proportion by alternation; that is, the first term is to the third as the second is to the fourth.

Let $a : b :: c : d$

To prove that $a : c :: b : d$

$$ad = bc$$

(The product of the extremes is equal to the product of the means.)

$$\therefore \quad a : c :: b : d$$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

4. State as an equality of two fractions. See p. 214.

PROPOSITION LXXX. THEOREM

In any proportion, the terms are in proportion by inversion; that is, the second term is to the first as the fourth term is to the third.

Let $a : b :: c : d$

To prove that $b : a :: d : c$

$$bc = ad$$

(The product of the extremes is equal to the product of the means.)

$$\frac{b}{a} = \frac{d}{c}$$

(We divide both members by ac.)

$\therefore$ $b : a :: d : c$

(We write in proportional form.)

Hence, the principle, Q.E.D.

5. State as an equality of two fractions. See p. 214.

PROPOSITION LXXXI. THEOREM

In any proportion, the terms are in proportion by composition; that is, the sum of the first two is to the first or second as the sum of the last two is to the corresponding term.

Let $a : b :: c : d$

To prove that $a + b : b :: c + d : d$

$$ad = bc$$

(The product of the extremes is equal to the product of the means.)

$$(a + b)d = (c + d)b$$

(We add bd to each member and factor.)

$\therefore$ $a + b : b :: c + d : d$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

6. State as an equality of two fractions. See p. 214.

PROPORTION — PRINCIPLES

PROPOSITION LXXXII. THEOREM

In any proportion, the terms are in proportion by division; that is, the difference of the first two is to the first or second as the difference of the last two is to the corresponding term.

Let $a : b :: c : d$

To prove that $a - b : b :: c - d : d$

$$ad = bc$$

$$(a - b)d = (c - d)b$$

(We subtract bd from each member.)

$\therefore$ $a - b : b :: c - d : d$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

PROPOSITION LXXXIII. THEOREM

In any proportion, the terms are in proportion by composition and division; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

Let $a : b :: c : d$

To prove that $a + b : a - b :: c + d : c - d$

$$\frac{a+b}{b} = \frac{c+d}{d} \qquad \frac{a-b}{b} = \frac{c-d}{d}$$

(By composition.) (By division.)

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

(We divide the equations member by member.)

$\therefore$ $a + b : a - b :: c + d : c - d$

(We write in proportional form.)

Hence, the principle, Q.E.D.

Proposition LXXXIV. Theorem

In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Let $$a:b=c:d=e:f$$

To prove that $$a+c+e:b+d+f::a:b$$

$$ab=ba$$

(The product of two quantities is the same in whatever order arranged.)

$$ad=bc \qquad af=be$$

(The product of the extremes is equal to the product of the means.)

$$(b+d+f)a=(a+c+e)b$$

(We add the three equations.)

$\therefore$ $$a+c+e:b+d+f::a:b$$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

Proposition LXXXV. Theorem

In any proportion, like powers or like roots of the terms are in proportion.

Let $$a:b::c:d$$

To prove that $$a^n:b^n::c^n:d^n$$

$$ad=bc$$

(The product of the extremes is equal to the product of the means.)

$$a^nd^n=b^nc^n$$

(We raise each member to the *n*th power.)

$\therefore$ $$a^n:b^n::c^n:d^n$$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

PROPORTION — PRINCIPLES

Proposition LXXXVI. Theorem

In two or more proportions, the products of the corresponding terms are in proportion.

Let $a : b :: c : d$

and $e : f :: g : h$

To prove that $ae : bf :: cg : dh$

$$ad = bc$$

$$eh = fg$$

(The product of the extremes is equal to the product of the means.)

$$ae \times dh = bf \times cg$$

(We multiply the equations member by member.)

$\therefore$ $ae : bf :: cg : dh$

(If the product of two quantities is equal to the product of two others, etc.)

Hence, the principle, Q.E.D.

Proposition LXXXVII. Theorem

Equimultiples of two quantities have the same ratio as the quantities themselves.

Let a, b, be the two quantities

and ma, mb, be the equimultiples

To prove that $a : b :: ma : mb$

$$\frac{a}{b} = \frac{a}{b}$$

(We write the ratio equal to itself.)

$$\frac{a}{b} = \frac{ma}{mb}$$

(We multiply both terms of the second ratio by m.)

$\therefore$ $a : b :: ma : mb$

(We write in proportional form.)

Hence, the principle, Q.E.D.

In a direct proof, the supposition is taken as a basis, and valid operations are performed until the desired conclusion is reached. In an indirect proof, the conclusion is taken as a basis, and valid operations are performed until some known result is reached. In the former process, since the supposition is true and since all the operations are valid, the conclusion must be true. In the latter, since the conclusion is true and since all the operations are valid, the supposition must be true. The one proceeds from the supposition to the conclusion; the other, from the conclusion to the supposition. The proofs thus far given are direct.

Prove indirectly:

7. Prop. LXXVI

8. Prop. LXXVII

9. Prop. LXXIX

10. Prop. LXXXI

11. Prop. LXXXII

12. Prop. LXXXIII

Ex. 7. Assume that $ad = bc$; dividing both members by bd, $\frac{a}{b} = \frac{c}{d}$; writing in proportional form, $a : b :: c : d$. Since the conclusion is true, and since the operations are valid, the supposition must be true, Q.E.D.

Ex. 10. Assume that $a + b : b :: c + d : d$; placing the product of the extremes equal to the product of the means, $ad + bd = bc + bd$; subtracting bd from both members, $ad = bc$. Since the conclusion is true, and since the operations are valid, the supposition must be true, Q.E.D.

13. If $a : b :: c : d$, prove directly that $3a : 2b :: 3c : 2d$.

Given $a : b :: c : d$; writing in fractional form, $\frac{a}{b} = \frac{c}{d}$; multiplying both members by $\frac{3}{2}$, $\frac{3a}{2b} = \frac{3c}{2d}$; writing in proportional form, $3a : 2b :: 3c : 2d$, Q.E.D.

14. If $a : b :: c : d$, prove indirectly that $3a : 2b :: 3c : 2d$.

Assume that $3a : 2b :: 3c : 2d$; putting product of extremes equal to product of means, $6ad = 6bc$; dividing by 6, $ad = bc$. Since the result is true and since the operations are valid, $3a : 2b :: 3c : 2d$, Q.E.D.

PROPORTION — PRINCIPLES

15. If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second; that is, if $a : b :: b : c$, then $a : c :: a^2 : b^2$. Prove by each method.

DIRECT PROOF

If $a : b = b : c$

$$\frac{a}{b} = \frac{b}{c}$$

(In fractional form.)

$$\frac{a}{b} = \frac{a}{b}$$

(An identity.)

$$\frac{a^2}{b^2} = \frac{a}{c}$$

(We multiply.)

$$\therefore\ a : c = a^2 : b^2$$

INDIRECT PROOF

If $a : c = a^2 : b^2$

$$ab^2 = a^2c$$

(Product of means equals product of extremes.)

$$b^2 = ac$$

(We divide by a.)

$$a : b = b : c$$

(If the product of two quantities, etc.)

$$\therefore\ a : c = a^2 : b^2$$

16. If four quantities are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second; that is, if $a : b = b : c = c : d$, then $a : d :: a^3 : b^3$. Prove by each method.

17. Prove that $a^2 - b^2 : a^2 - 3ab :: c^2 - d^2 : c^2 - 3cd$ if $a : b :: c : d$.

18. Prove that $a^2 + ab + b^2 : a^2 - ab + b^2 :: c^2 + cd + d^2 : c^2 - cd + d^2$, if $a : b :: c : d$.

19. Find the mean proportional to $12ax^2$ and $3a^3$.

20. Which is the greater ratio and by how much, $3 : 4$ or $3^2 : 4^2$?

21. Find a fourth proportional to x^3, xy, and $5x^2y$.

22. Find a third proportional to $\frac{x}{y} + \frac{y}{x}$ and $\frac{x}{y}$.

23. What is the test of the correctness of a given proportion?

PROPORTION — SOLUTIONS

In the solution of equations, the first impulse is to place the product of the extremes equal to the product of the means and to simplify the result. This impulse should be resisted because the process is often laborious. In every case, the example should be carefully inspected and the plan of attack formed before anything is written.

24. Find the value of x in the proportion,

$$a+\sqrt{x} : a-\sqrt{x} :: b+\sqrt{c} : b-\sqrt{c}.$$

By composition and division, $2a : 2\sqrt{x} :: 2b : 2\sqrt{c}$; dividing by 2, $a : \sqrt{x} :: b : \sqrt{c}$; squaring, $a^2 : x :: b^2 : c$; $x = \frac{a^2c}{b^2}$.

25. Find the value of x in the proportion,

$$x^2-4 : x^2-9 :: x^2-5x+6 : x^2+4x+3.$$

Expressing the products of the extremes and means and factoring, $(x+2)(x-2)(x+3)(x+1) = (x+3)(x-3)(x-2)(x-3)$; dividing by the common factors, $x-2$ and $x+3$, $(x+2)(x+1) = (x-3)^2$; $x = \frac{7}{9}$. Also, $x-2=0$ and $x+3=0$; $x = 2, -3$. See p. 189.

Find the value of x:

26. $x+1 : x+4 :: 2x-1 : x+6$

27. $x+a : 2x-b :: 3x+b : 4x-a$

28. $\sqrt{x}+\sqrt{b} : \sqrt{x}-\sqrt{b} :: a : b$

29. $x^2-a^2 : x+a :: x+a : 2$

30. $\sqrt[3]{x+a} : \sqrt[3]{x-a} = m : n$

31. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} = c$

32. $\dfrac{x^2+x-2}{x-2} = \dfrac{4x^2+5x-6}{5x-6}$

33. $\sqrt{x^2-a^2} : \sqrt{x-a} :: \sqrt{x+a} : x\sqrt{x}$

34. $x+\sqrt{1-x^2} : x-\sqrt{1-x^2} = a+\sqrt{b^2-a^2} : a-\sqrt{b^2-a^2}$

PROPORTION — PROBLEMS

Problems involving multiplication and division can be expressed in terms of ratio or proportion, and *vice versa.*

1. (*a*) Three times A's age is 2 times B's age; in 10 years, 4 times A's age will be 3 times B's age. Find the age of each. State in terms of proportion.

A's age is to B's age as 2 is to 3; in 10 years A's age will be to B's age as 3 is to 4. Find the age of each.

(*b*) State in terms of ratio.

The ratio of A's age to B's is $\frac{2}{3}$; in 10 years, the ratio of their ages will be $\frac{3}{4}$. Find the age of each.

2. B has 3 times as many peaches as A. If A has 5 peaches, how many has B? State in terms of a ratio.

The ratio of B's peaches to A's is 3. If A has 5 peaches, how many has B?

It follows that most of the preceding problems may be solved in the phraseology of proportion, and that those in this exercise may be solved in terms of multiplication and division.

3. The length of a room is to its width as 4 to 3, and the area of the floor is 588 sq. ft. What are the dimensions of the room?

4. Find two numbers such that the greater is to the less as their sum is to 6, and the greater is to the less as their difference is to 2.

5. What two numbers are those whose sum is to their difference as 7 is to 1, and whose product is to the sum as 24 to 7?

6. The length of a rectangular field is to its width as 5 is to 4, but if 4 rods be added to the length and 5 rods to the width, they will be to each other as 6 is to 5. Find the area.

7. At simple interest, in what time will a dollars at r per cent gain as much as b dollars in t years at n per cent?

8. In a mile race between a bicycle and a tricycle their rates were as 5 to 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rates of each.

Let $5x$ and $4x$ equal yards per minute of each. $\frac{1760}{5x}$ = minutes for bicycle to go a mile; $\frac{1760}{4x}$ = minutes for tricycle; $\frac{176}{4x}$ = minutes by which tricycle was beaten; $\frac{1760}{4x} - \left(\frac{1}{2} + \frac{176}{4x}\right) = \frac{1760}{5x}$.

9. The product of two numbers is 15, and the cube of their sum is to the sum of their cubes as 64 is to 19. Find the numbers.

$x^3 + 3x^2y + 3xy^2 + y^3 : x^3 + y^3 :: 64 : 19$; whence, $3xy(x+y) : x^3 + y^3 :: 45 : 19$; whence, $3xy : x^2 - xy + y^2 :: 45 : 19$; and so on.

10. The product of two numbers is 10, and the difference of their cubes is to the cube of their difference as 13 : 3. Find the numbers.

11. Two wagons with their loads have their weights in the ratio of 4 to 5; parts of their loads in the ratio of 6 to 7 being removed, they weigh in the ratio of 2 to 3, and the sum of their weights is then 10 tons. What were the weights at first?

Let $4x$ and $5x$ = loads at first; $6y$ and $7y$ = parts removed.

12. A and B are in partnership; A's capital is to the whole capital as 5 to 8; but if A withdraws \$2000 and B adds \$2000, A's capital will be to the whole as 3 to 5. Required each man's share of the stock.

13. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

14. Find four proportionals such that the sum of the extremes is 21, the sum of the means is 19, and the sum of the squares of all four numbers is 442.

VARIATION

PRINCIPLES

If A, B, and C are three variables and m is a constant, the ratio of A to one or to both of the other variables combined may be expressed in four general forms:

$A:B=m$, or $A \propto B$, read, A varies directly as B.

$A:\frac{1}{B}=m$, or $A \propto \frac{1}{B}$, read, A varies inversely as B.

$A:BC=m$, or $A \propto BC$, read, A varies jointly as B and C.

$A:\frac{B}{C}=m$, or $A \propto \frac{B}{C}$, read, A varies directly as B and inversely as C.

1. Illustrate the meaning of $A \propto B$, or $A:B=m$.

The amount of work varies directly as the number of men; that is, if a work can be done by a number of men, m times the work will require m times the number of men. If A is the amount of work and B the number of men, $A \propto B$, or $A:B=m$.

2. Illustrate the meaning of $A \propto BC$, or $A:BC=m$.

The amount of work varies directly as the number of men and directly as the number of days; that is, if a work can be done by a number of men in a number of days, mn times the work can be done by m times the number of men in n times the number of days. If A is the amount of work, B the number of men, and C the number of days, $A \propto BC$, or $A:BC=m$.

3. Illustrate the meaning of $A \propto \frac{1}{B}$, or $A:\frac{1}{B}=m$.

The number of men varies inversely as the number of days; that is, if a number of men can do a work in a number of days, m times the number of

men will require $\frac{1}{m}$ times the number of days. If A is the number of men and B the number of days, $A \propto \frac{1}{B}$, or $A : \frac{1}{B} = m$.

4. Illustrate the meaning of $A \propto \frac{B}{C}$, or $A : \frac{B}{C} = m$.

The number of men varies directly as the work and indirectly as the number of days; that is, if a number of men can do a work in a number of days, mn times the number of men can do m times the work in $\frac{1}{n}$ times the number of days.

5. The area of a circle equals πr^2. How does the area of a circle vary?

Let $A =$ area; then $A = \pi r^2$ and $\frac{A}{r^2} = \pi$. Since π is a constant, the area of a circle varies as the square of its radius.

6. The volume of a sphere equals $\frac{4}{3}\pi r^3$. How does the volume of a sphere vary?

7. The volume of a cone equals π times the square of the radius of its base times one third of its altitude. How does the volume of a cone vary?

8. How does the distance passed by a wagon vary in terms of miles per hour?

9. How does the distance passed by a wagon vary in terms of hours?

10. How does the distance passed by a wagon vary in terms of hours and miles per hour?

11. If the illumination of an object at a distance of 1 ft. from the source of light is m^2 times the illumination at a distance of m ft., how does the illumination vary?

12. If the force of gravity at a distance of 1 ft. is m^2 times the force at a distance of m ft., and is n times as great for n lb. as for 1 lb., how does the force of gravity vary in terms of distance and weight?

SOLUTION OF PROBLEMS

To solve a problem in variation, it is necessary to find the general relation and to substitute two sets of values for the variables. Each substitution will give an expression for the constant, and these expressions may be placed equal to each other.

1. If $x \propto \frac{y}{z}$, and $x = 12$ when $y = 2$ and $z = 3$, find x when $y = 2$ and $z = 9$.

$$x : \frac{y}{z} = m \qquad\qquad x = 12 \times \frac{2}{9} \times \frac{3}{2}$$

$$12 : \frac{2}{3} = x : \frac{2}{9} \qquad\qquad x = 4$$

Substituting the first set of values, $12 : \frac{2}{3} = m$; substituting the second set of values, $x : \frac{2}{9} = m$; equating these expressions, $12 : \frac{2}{3} = x : \frac{2}{9}$. It is simpler, however, to equate the values as above without writing m twice.

2. If $x \propto y$, and $x = 12$ when $y = 3$, what is the value of x when $y = 5$?

3. If $x \propto y$, and $x = 6$ when $y = 2\frac{1}{2}$, what will x equal when $y = 4\frac{3}{4}$?

4. If $x \propto \frac{1}{y}$, and $y = 4$ when $x = 15$, find y when $x = 6$.

5. A varies jointly as B and C, and $A = 6$ when $B = 3$ and $C = 2$. Find A when $B = 5$ and $C = 7$.

6. If A varies inversely as B, and when $A = 2$ the corresponding value of B is 36, find the corresponding value of B when $A = 9$.

7. If the square of x varies as the cube of y, and $x = 3$ when $y = 4$, find the equation between x and y.

Ex. 5.

$$A : BC = m$$

$$6 : 6 = A : 35$$

$$A = 35$$

Writing A varies jointly as B and C in different form, $A : BC = m$; substituting both sets of values, $6 : 6 = A : 35$; $A = 35$.

Every problem involving ratio, and hence every problem involving multiplication or division, may be solved by variation.

8. If 1 apple costs 2¢, how much will 3 apples cost? Solve by variation.

We must find the relation of a number of apples to their cost. The number of apples varies directly as their cost. If $N =$ the number and $C =$ their cost, $N \propto C$. Then we can solve in the usual manner.

$$N : C = m, \qquad 1 : 2 = 3 : C, \qquad C = 6.$$

9. If 3 men can do a piece of work in 6 days, in how many days can 9 men do it? Solve by variation.

RELATION. The number of men varies inversely as the number of days.

10. If 3 men can do a piece of work in 6 days of 8 hours each, how many days of 9 hours each will be required by 2 men?

Let $M =$ number of men, $D =$ number of days, $H =$ number of hours. Then $M \propto \frac{1}{DH}$.

11. (*a*) A sphere is equivalent in volume to the sum of three spheres whose radii are 3 inches, 4 inches, and 5 inches. Find the radius of the large sphere. Solve by variation.

Let $v =$ the volume and $R =$ the radius of the sphere, and let v_1, v_2, v_3 = the special values. We must find the relation of v to R. This is, $v \propto R^3$.

$$\begin{array}{llr}
v : R^3 = m & & (1) \\
v_1 : 3^3 = m, & \text{or } v_1 = & 27\,m \\
v_2 : 4^3 = m, & \text{or } v_2 = & 64\,m \\
v_3 : 5^3 = m, & \text{or } v_3 = & 125\,m \\ \hline
\text{Adding,} & v = & 216\,m
\end{array}
\qquad
\begin{array}{l}
216\,m : R^3 = m \\
\dfrac{216\,m}{R^3} = m \\
R^3 = 216 \\
R = 6
\end{array}$$

Since $v_1 : 3^3 = m$, $v_1 = 27\,m$; since $v_2 : 4^3 = m$, $v_2 = 64\,m$; since $v_3 : 5^3 = m$, $v_3 = 125\,m$; their sum $= 216\,m$, or $v = 216\,m$; substituting this value of v in (1), $216\,m : R^3 = m$, or $R = 6$ inches.

NOTE. v_1, v_2, and v_3 are used as different letters. They are read v sub one, v sub two, and v sub three.

SOLUTION OF PROBLEMS

(*b*) Solve by proportion.

From geometry, we know that the volumes of two spheres are to each other as the cubes of their radii.

$$v : v_1 :: R^3 : 27, \qquad \text{or } v_1 = \frac{27\,v}{R^3}$$

$$v : v_2 :: R^3 : 64, \qquad \text{or } v_2 = \frac{64\,v}{R^3}$$

$$v : v_3 :: R^3 : 125, \qquad \text{or } v_3 = \frac{125\,v}{R^3}$$

Adding,

$$v = \frac{216\,v}{R^3}$$

$$vR^3 = 216\,v$$

$$R^3 = 216$$

$$R = 6$$

(*c*) Solve by the usual algebraic method.

From geometry, we know that the volume of a sphere $= \frac{4}{3}\pi R^3$.

$$v_1 = \frac{4}{3} \times 27\,\pi \qquad\qquad v = \frac{4}{3}\pi R^3$$

$$v_2 = \frac{4}{3} \times 64\,\pi \qquad\qquad \frac{4}{3}\pi R^3 = \frac{4}{3} \times 216\,\pi$$

$$v_3 = \frac{4}{3} \times 125\,\pi \qquad\qquad R^3 = 216$$

$$v = \frac{4}{3} \times 216\,\pi \qquad\qquad R = 6$$

The remaining problems of this exercise may be solved in different ways, but the pupil should solve them by variation.

12. If 8 men earn \$400 in 5 weeks, find how many weeks it will take 6 men to earn \$240.

RELATION. The number of men varies directly as the number of dollars earned and indirectly as the number of weeks.

13. If 8 men earn \$400 in 5 weeks, find how many men will earn \$240 in 4 weeks.

VARIATION

14. The area of a circle is equivalent to the sum of the areas of two circles whose radii are 3 in. and 4 in. Find the radius.

15. The volume of a cylinder is equivalent to the sum of the volumes of three cylinders whose radii are 6 in., 8 in., and 10 in.; the altitudes of the four are equal. Find the radius.

16. If the volume of a pyramid varies jointly as the area of its base and altitude, what will be the altitude of a pyramid whose base is 12, equivalent to the sum of two pyramids whose bases are 5 and 8, and altitude 12 and 6 respectively?

17. If the illumination from a source of light varies inversely as the square of the distance, how much farther from a candle must a book, which is now 15 in. off, be removed so as to receive just $\frac{1}{8}$ as much light?

18. The distance through which a body falls from rest varies as the square of the time it falls. If a body falls 64 ft. in 2 seconds, how far does it fall in 6 seconds?

19. If a body falls 576 ft. in 6 seconds, how far does it fall in 2 seconds?

20. The velocity with which a liquid escapes from an orifice varies as the square root of the depth of the liquid above the orifice; when the depth is 402 inches, the velocity is 160.8 inches. What is the velocity when the depth is 16.08 inches?

21. The square of the time of a planet's revolution varies as the cube of its distance from the sun. If the distances of the Earth and Mercury from the sun are 91 and 35 millions of miles, find in days the time of Mercury's revolution.

22. The volume of a gas varies directly as the temperature and inversely as the pressure; when the pressure is 15 lb. and the temperature 260° the volume is 200 cu. in. Find the volume when the pressure is 18 lb. and the temperature 390°.

ARITHMETICAL PROGRESSION

DEVELOPMENT

Quantities may increase or decrease by a common difference. The common difference is found by subtracting any term from that which immediately follows it.

$$1,\ 3,\ 5,\ 7,\ \cdots$$

$$8,\ 3,\ -2,\ -7,\ \cdots$$

$$a,\ a+d,\ a+2\,d,\ \cdots$$

Each of the above is a series in arithmetical progression. In the first, the common difference is 2; in the second, -5; in the third, d.

Examining these series, we see that we may find the first term, a; the last term, l; the number of terms, n; the common difference, d; and the sum of the terms, s.

PROPOSITION LXXXVIII. THEOREM

$$l = a + (n-1)\,d$$

In the general arithmetical series, a, $a+d$, $a+2\,d$, $a+3\,d$, $a+4\,d$, $\cdots$,

$a =$ 1st term	$a + 3\,d =$ 4th term
$a + d =$ 2d term	. . . .
$a + 2\,d =$ 3d term	$a + (n-1)d = n$th term

Examining these, we see that each term is equal to a, plus d with a coefficient one less than the number of the term.

Hence, the principle, Q.E.D.

PRINCIPLES

Proposition LXXXIX. Theorem

$$s = \frac{n}{2}(a + l)$$

In the general arithmetical series,

$$a,\ a + d,\ a + 2d,\ \cdots,$$

$$s = a + (a + d) + (a + 2d) + \cdots + l \qquad \text{By definition}$$

$$s = l + (l - d) + (l - 2d) + \cdots + a \qquad \text{By definition}$$

Adding, $2s = (a + l) + (a + l) + (a + l) + \cdots n \text{ times}$

$$= n(a + l)$$

$\therefore$ $s = \frac{n}{2}(a + l)$

Hence, the principle, Q.E.D.

Proposition XC. Theorem

From the formulæ for l and s, all the terms of an arithmetical progression may be found provided any three of them are given.

Given $l = a + (n - 1)d$ (1)

and $s = \frac{n}{2}(a + l)$ (2)

If three of the four letters in (1) or if three of the four letters in (2) are known, there will result one equation with one unknown quantity.

If three of the five letters in both (1) and (2) are known, there will result two equations with two unknown quantities.

Hence, the principle, Q.E.D.

Note. The pupil should memorize these two formulæ.

SOLUTIONS

To solve examples in arithmetical progression, it is best to substitute the given terms in one or both the formulæ and to simplify.

Find the value of:

1. l; given $a=1$, $d=2$, $n=7$
2. a; given $l=13$, $n=7$, $d=2$
3. n; given $l=13$, $a=1$, $d=2$
4. d; given $l=13$, $a=1$, $n=7$
5. a in terms of l, n, d
6. n in terms of l, a, d
7. d in terms of l, a, n
8. l in terms of a, n, d

Ex. 2. $l = a + (n-1)d$; substituting, $13 = a + (7-1)2$, $a = 1$.

Find the value of:

9. s; given $n=6$, $a=7$, $l=22$
10. n; given $s=87$, $a=7$, $l=22$
11. a; given $s=87$, $n=6$, $l=22$
12. l; given $s=87$, $n=6$, $a=7$
13. n in terms of s, a, l
14. a in terms of s, n, l
15. l in terms of s, n, a
16. s in terms of n, a, l

Ex. 11. $s = \frac{n}{2}(a+l)$; substituting, $87 = \frac{6}{2}(a+22)$, $a = 7$

Find the value of:

17. d; $l=28$, $a=3$, $s=93$
18. d; $l=28$, $n=6$, $s=93$
19. d; $a=3$, $n=6$, $s=93$
20. s; $l=28$, $a=3$, $d=5$
21. s; $l=28$, $n=6$, $d=5$
22. s; $a=3$, $n=6$, $d=5$
23. d; given l, a, s
24. s; given l, a, d

Ex. 19. $l = a + (n-1)d \quad (1)$

$s = \frac{n}{2}(a+l) \quad (2)$

Substituting, $l = 3 + (6-1)d \quad (1)$

$93 = \frac{6}{2}(3+l) \quad (2)$

Find the last term and the sum of the series:

25. $2, 5, 8, 11, \cdots\cdots$ to 8 terms

26. $2a - b,\ 4a - 3b,\ 6a - 5b, \cdots$ to n terms

27. $x,\ 3x,\ 5x, \cdots\cdots$ to n terms

28. $1,\ \frac{6}{7},\ \frac{5}{7}, \cdots\cdots$ to 1000 terms

29. $\frac{1}{4},\ -\frac{1}{4},\ -\frac{3}{4}, \cdots\cdots$ to 8 terms

30. $\frac{n-1}{n},\ \frac{n-2}{n},\ \frac{n-3}{n}, \cdots\cdots$ to n terms

Given:

31. $a = -2,\ d = 4,\ s = 160$; find n

32. $l = 11,\ d = 3,\ n = 7$; find s

33. $s = 200,\ n = 12,\ d = 5$; find a

34. $d = -3,\ l = -39,\ s = -264$; find n

35. $n = 8,\ a = 8,\ d = 5$; find l

36. $d = \frac{3}{2},\ s = 58,\ a = 2$; find n

Insert:

37. 3 arithmetical means between 14 and 16

38. 6 arithmetical means between 5 and 40

39. 4 arithmetical means between 12 and 32

40. 3 arithmetical means between 1 and 19

41. 2 arithmetical means between $5a - 6b$ and $5b - 6a$

42. The arithmetical mean between $(a+b)^2$ and $(a-b)^2$

SOLUTION OF PROBLEMS

In most problems it is best to let x equal the first term and y the common difference, but in some cases the equations are more readily solved by the use of the following expressions:

$$\cdots x-2y,\ x-y,\ x,\ x+y,\ x+2y\cdots \text{ odd number}$$
$$\cdots x-3y,\ x-y,\quad x+y,\ x+3y\cdots \text{ even number}$$

In a series of an odd number of terms, x is the middle term and y the common difference; in a series of an even number of terms, $x+y$ and $x-y$ are the middle terms and $2y$ is the common difference.

1. (*a*) There are 3 numbers in arithmetical progression whose sum is 24, and the sum of whose squares is 224. Find the numbers. Let $x-y$, x, $x+y=$ the numbers.

(*b*) Solve by letting x equal the first number and y the common difference. Why is it better to use the method of (*a*)?

2. Find four numbers in arithmetical progression whose sum is 40, and the sum of whose squares is 480. Let $x-3y$, $x-y$, $x+y$, $x+3y=$ the numbers.

3. The sum of 5 numbers in arithmetical progression is 45, and the product of the first and fifth terms is $\frac{5}{8}$ of the product of the second and fourth. Find the numbers.

4. Divide 20 into 4 parts which are in arithmetical progression, such that the product of the first and fourth is to the product of the second and third as $2:3$.

5. Find the series in which the 27th term is 186 and the 45th term is 312. Let $x=$ first term and y the common difference.

6. The 8th term of an arithmetical progression is 17 and the 13th term is 27. Find the 10th term.

7. The sum of 7 terms of an arithmetical progression is 63; the sum of 15 terms, 255. Find the series. Let $x=$ first term and y the common difference.

8. The sum of m terms of an arithmetical progression is m^2+2m; the sum of n terms, n^2+2n. Find the series.

9. The base of a right triangle is 12 inches and its sides are in arithmetical progression. Find the other sides.

10. A saves every year \$25, which he puts at interest at the rate of 4 per cent a year. How long will it take for the interest to amount to \$91?

11. A person received a gift of \$100 per year from his birth until he was 21 years old; these sums were deposited in a bank and drew simple interest at 6%. How much was due him when he became of age?

12. Find the difference between the sum of the even numbers and the sum of the odd numbers from 1 to 200 inclusive.

13. The sum of n terms of the series 3, 5, 7 $\cdots$ exceeds the nth term by 63. Find the value of n.

14. Find the series of which the mth term is $m-n+mn$, and the nth term $m-n+n^2$.

15. A man increased his capital stock \$500 at the end of each year for 10 years, and then had invested \$6500. What was his capital at first?

16. The digits of a number of three figures are in arithmetical progression; hundreds' digit exceeds the sum of tens' and units' by 1; and if 594 is subtracted from the number, the digits will occur in reverse order. Find the number.

17. A travels uniformly 20 miles a day. B starts 3 days later from the same place and travels in the same direction, 8 miles the first day, 12 miles the second, and so on in arithmetical progression. In how many days will B overtake A?

18. A and B set out at the same time to meet each other from two places 118 miles apart; their daily journeys are in arithmetical progression, A's increase is 2 miles each day and B's is 5 miles each day; on the day at the end of which they meet, each traveled exactly 20 miles. Find the duration of the journey.

GEOMETRICAL PROGRESSION

DEVELOPMENT

Quantities may increase or decrease by a common ratio. The common ratio is found by dividing any term by that which immediately precedes it.

$$2,\ 4,\ 8,\ 16,\ \ldots$$

$$9,\ 3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \ldots$$

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

Each of the above is a series in geometrical progression. In the first the common ratio is 2; in the second, $\frac{1}{3}$; in the third, r.

Examining these series, we see that we may find the first term, a; the last term, l; the number of terms, n; the common ratio, r; and the sum of the terms, s.

Proposition XCI. Theorem

$$l = ar^{n-1}$$

In the general geometrical series, $a, ar, ar^2, ar^3, ar^4, \ldots$

$a = 1\text{st term}$	$ar^3 = 4\text{th term}$
$ar = 2\text{d term}$	. . . . . .
$ar^2 = 3\text{d term}$	$ar^{n-1} = n\text{th term}$

Examining these, we see that each term is equal to a, multiplied by r with an exponent one less than the number of the term.

Hence, the principle, Q.E.D.

PRINCIPLES

Proposition XCII. Theorem

$$s = \frac{rl - a}{r - 1}$$

In the general geometrical series,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \cdots$$

$s = a + ar + ar^2 + \cdots + ar^{n-2} + l$ By definition

Multiplying by r, $rs = ar + ar^2 + \cdots + ar^{n-2} + l + rl$

Subtracting, $s(r - 1) = rl - a$

$\therefore$ $s = \dfrac{rl - a}{r - 1}$

Hence, the principle, Q.E.D.

NOTE. $ar^{n-2} \times r = ar^{n-1}$; $ar^{n-1} = l$. Hence, in the product, l is the next term after ar^{n-2}.

Proposition XCIII. Theorem

From the formulæ for l and s, all the terms of a geometrical progression may be found provided any three of them are given.

Given $l = ar^{n-1}$ (1)

and $s = \dfrac{rl - a}{r - 1}$ (2)

If three of the four letters in (1) or if three of the four letters in (2) are known, there will result one equation with one unknown quantity.

If three of the five letters in both (1) and (2) are known, there will result two equations with two unknown quantities.

Hence, the principle, Q.E.D.

NOTE. The pupil should memorize these two formulæ.

SOLUTIONS

To solve examples in geometrical progression, it is best to substitute the given terms in one or both the equations and to simplify.

Find the value of:

1. l; given $a = 2$, $r = 3$, $n = 4$
2. a; given $l = 54$, $r = 3$, $n = 4$
3. n; given $l = 54$, $a = 2$, $r = 3$
4. r; given $l = 54$, $a = 2$, $n = 4$
5. a in terms of l, r, n
6. n in terms of l, a, r
7. r in terms of l, a, n
8. l in terms of a, r, n

Ex. 3. $l = ar^{n-1}$; substituting, $54 = 2 \times 3^{n-1}$, $(n-1) \log 3 = \log 27$, $n - 1 = \log 27 \div \log 3 = 1.4314 \div .4771 = 3$; $n = 4$

Find the value of:

9. s; given $r = 2$, $l = 40$, $a = 5$
10. r; given $s = 75$, $l = 40$, $a = 5$
11. a; given $s = 75$, $r = 2$, $l = 40$
12. l; given $s = 75$, $r = 2$, $a = 5$
13. r in terms of s, l, a
14. a in terms of s, r, l
15. l in terms of s, r, a
16. s in terms of r, l, a

Ex. 12. $s = \dfrac{rl - a}{r - 1}$; substituting, $75 = \dfrac{2l - 5}{2 - 1}$, $l = 40$

Find the value of:

17. s; given $l = 27$, $a = 1$, $n = 4$
18. s; given $l = 27$, $a = 1$, $r = 3$
19. s; given $a = 1$, $r = 3$, $n = 4$
20. n; given $l = 27$, $a = 1$, $s = 40$
21. n; $a = 1$, $r = 3$, $s = 40$
22. n; $l = 27$, $a = 1$, $r = 3$
23. n; given l, a, s
24. s; given l, a, n

Ex. 21. $l = ar^{n-1}$ (1) Substituting, $l = 1 \times 3^{n-1}$ (1)

$s = \dfrac{rl - a}{r - 1}$ (2) $40 = \dfrac{3l - 1}{3 - 1}$ (2)

Find the last term and the sum of the series:

25. $3,\ 9,\ 27,\ \ldots\ldots\ldots$ to 8 terms

26. $3,\ 1,\ \frac{1}{3},\ \ldots\ldots\ldots$ to 7 terms

27. $.1,\ .5,\ 2.5,\ \ldots\ldots\ldots$ to 8 terms

28. $a,\ \frac{a}{r},\ \frac{a}{r^2},\ \ldots\ldots\ldots$ to n terms

29. $\sqrt{2},\ \sqrt{6},\ 3\sqrt{2},\ \ldots\ldots\ldots$ to 12 terms

30. $\frac{3}{5},\ \frac{2}{5},\ \frac{4}{15},\ \ldots\ldots\ldots$ to 4 terms

Given:

31. $a = 2,\quad l = 162,\quad n = 5$; find r

32. $a = 2,\quad l = 10\frac{1}{8},\quad n = 5$; find s

33. $r = 2,\quad n = 10,\quad s = 5115$; find l

34. $n = 3,\quad l = 100,\quad r = 5$; find a

35. $a = 64,\quad n = 6,\quad l = 2$; find s

36. $s = 728,\quad r = 3,\quad l = 486$; find n

Insert:

37. 2 geometrical means between 8 and 125

38. 2 geometrical means between 8 and -1

39. 4 geometrical means between 160 and 5

40. 5 geometrical means between 3 and 192

41. 2 geometrical means between a and $a(a+b)^3$

42. The geometrical mean between $18\,x^3y$ and $2\,xy^3$

SOLUTION OF PROBLEMS

In most problems, it is best to let x equal the first term and y the ratio, but in some cases the equations are more readily solved by the use of the following expressions:

$x^2,\ xy,\ y^2$	3 terms
$x^3,\ x^2y,\ xy^2,\ y^3$	4 terms
$x^4,\ x^3y,\ x^2y^2,\ xy^3,\ y^4$	5 terms

In these, $\frac{y}{x}$ is the ratio and x^2, x^3, x^4, the first term.

1. (*a*) The sum of three numbers in geometrical progression is 26; the sum of their squares is 364. Find the numbers. Let $x = a$ and $y = r$.

(*b*) Solve by letting x^2, xy, y^2 equal the numbers. Which do you prefer?

2. (*a*) The sum of four numbers in geometrical progression is 15; the sum of the first and last exceeds the sum of the other two by 3. Find the numbers. Let $x = a$ and $y = r$.

(*b*) Solve by letting x^3, x^2y, xy^2, y^3 equal the numbers. Which do you prefer?

3. Find three numbers in geometrical progression; the product of the first two is 75; the product of the last two is 675.

4. Find four numbers in geometrical progression the sum of whose first and third terms is 40, and the sum of whose second and fourth terms is 120.

5. There are three numbers in arithmetical progression whose sum is 27; if the first is multiplied by 4, the second by 2, and the third by $1\frac{4}{5}$, the results form a geometrical progression. What are the numbers?

6. There are three numbers in geometrical progression whose sum is 57; if the first is multiplied by $\frac{1}{4}$, the second by $\frac{1}{2}$, and the third by $\frac{5}{9}$, the results form an arithmetical progression. What are the numbers?

ANNUITIES, SINKING FUNDS, ETC.

A sinking fund is a sum of money set apart annually for a special purpose.

Thus, a county owes \$50,000 and sets apart annually \$6000 to pay this debt; \$6000 is a sinking fund.

An annuity is a fixed sum of money, payable annually for a number of years.

Thus, a man receives a yearly pension of \$500; \$500 is an annuity.

In the case of annuities and kindred problems, it is customary to compute compound interest. A knowledge of compound interest, geometrical progression, and logarithms is necessary.

PROPOSITION XCIV. AXIOM

At the close of the transaction, the sum of the amounts of the payments must equal the sum of the amounts of the receipts.

7. A county owes \$50,000. What sum set apart annually as a sinking fund will pay the debt in 6 years, if money is worth 5%?

The close of the transaction is at the end of 6 years. It is necessary to equate the sum of the amounts of the payments (\$6000 at the end of each year) and the sum of the amounts of the receipts (\$50,000).

Let x = number of dollars set aside annually, 9846.46

$x(1.05)^5$ = amount of 1st reserve at end of 6 yr.

$x(1.05)^4$ = amount of 2d reserve at end of 6 yr.

- - - - - - - - - - - - - - - - - - - -

x = amount of 6th reserve at end of 6 yr.

Since these 6 terms form a geometrical progression in which $a=x$, $r=1.05$, $n=6$, $l=x(1.05)^5$, we can find their sum from the formula, $s=\frac{rl-a}{r-1}$. Substituting,

$$s=\frac{x(1.05)^6-x}{.05}=\frac{x(1.05^6-1)}{.05} \qquad (1)$$

ANNUITIES, SINKING FUNDS, ETC.

The amount of the debt at the end of 6 yr. is $\$50,000(1.05)^6$.

Equating, $$\frac{x(1.05^6 - 1)}{.05} = 50,000(1.05)^6$$

Clearing, $$x(1.05^6 - 1) = 2500(1.05)^6$$

$$x = \frac{2500(1.05)^6}{1.05^6 - 1} \qquad (2)$$

We will now find by logarithms the value of 1.05^6, substitute in (2), and simplify.

$$\log 1.05^6 = 6 \log 1.05 \qquad x = \frac{2500 \times 1.3403}{.3403}$$

$$= .1272$$

$$1.05^6 = 1.3403 \qquad x = 9846.46$$

8. What annual appropriation as a sinking fund must a school district make to pay a debt of \$15,000 due in 17 yr., money worth 4% compound interest?

9. What sum should be paid for an annuity of \$100 a year, to be paid for 40 years, money being worth 4% per annum?

The present worth is the sum that put at interest to-day will amount to the same as the annuities at the end of 40 yr.; *i.e.* at the end of 40 yr. the sum of the amounts of the payments (\$100 at the end of each year) must equal the sum of the amounts of the receipts (the present worth).

Let $x =$ the present worth; $x(1.04)^{40} =$ the sum of the amounts of the receipts; $\dfrac{100(1.04)^{40} - 100}{.04} =$ the sum of the amounts of the payments.

10. What is the present worth of an annuity of \$1127 to continue 3 years, allowing 7% compound interest?

11. What should be paid for a perpetual annuity of \$200, interest at 2%?

A perpetual annuity is an annuity that is to be paid annually forever. The present worth is that sum which will gain \$200 in one year at 2%.

12. (*a*) A person borrows \$5225. How much must he pay in equal annual instalments in order that the whole debt may be discharged in 12 years, allowing 4 per cent compound interest? Solve by the United States rule for partial payments.

$$5225\,(1.04) - x = \text{end of first year}$$
$$5225\,(1.04)^2 - 1.04\,x - x = \text{end of second year}$$
$$5225\,(1.04)^3 - 1.04^2x - 1.04\,x - x = \text{end of third year}$$
$$5225\,(1.04)^{12} - (1.04)^{11}x - \cdots - 1.04\,x - x = \text{end of 12 years} = 0$$
$$\frac{x\,(1.04)^{12} - x}{.04} = 5225\,(1.04)^{12}$$

(*b*) Solve by the principle that the sum of the amounts of the payments must equal the sum of the amounts of the receipts.

$x\,(1.04)^{11}$ = amount of first payment; $x\,(1.04)^{10}$ = amount of second payment; $\cdots$; x = amount of twelfth payment.

13. (*a*) A man bought a house for \$5500, paying \$1000 cash and \$50 at the end of each month until principal and interest at 5% were fully paid. In how many months did he own the property?

Let x=number of months. $50(1.00\frac{5}{12})^{x-1}$=amount of first monthly payment at end of x months; $50(1.00\frac{5}{12})^{x-2}$=amount of second; $\cdots$; 50=amount of last.

$$\frac{50\,(1.00\frac{5}{12})^x - 50}{.00\frac{5}{12}} = 4500\,(1.00\tfrac{5}{12})^x$$

(*b*) His taxes were \$80 a year and other expenses for insurance, repairs, etc., \$38 a year. How much did he invest each month in the property?

Ans. \$64; count the \$50 that he might have received if he had put the \$1000 at interest.

(*c*) If he had rented the house for \$40 a month, had put the \$1000 at interest, and had invested \$24 at the end of each month at 5%, in how many months would he have had \$5500, or enough to buy the house? Count compound interest at the end of each month.

IMAGINARIES

DEVELOPMENT

Rational numbers, surds, and imaginaries (pp. 132 and 171) may result from the solution of the equation, $x^n = a$. If a is a perfect power of the nth degree, x becomes a rational number; if a is not a perfect power of the nth degree, x becomes a surd; if n is even and a is negative, x becomes imaginary. The value of a rational number can be expressed as an integer or as a fraction; the value of a surd can be expressed in fractional form true to any degree of approximation but never exactly; the value of an imaginary cannot be expressed by any method thus far discussed without the use of an even root of a negative number. Thus,

If $x^2 = 4$, $x = \sqrt{4} = +2$, rational.

If $x^2 = 2$, $x = \sqrt{2} =$ (approx.) $1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \cdots$ surd.

If $x^2 = -2$, $x = \sqrt{-2} =$ (approx.) $1\sqrt{-1}, \frac{14}{10}\sqrt{-1}, \frac{141}{100}\sqrt{-1}, \cdots$ imaginary.

Imaginaries may be added and subtracted, multiplied and divided, raised to powers and depressed to roots, and treated in every way that rationals and surds may be treated because they result from valid operations upon real numbers.

For the sake of simplicity, it is customary to regard the even root of a number as plus when the true sign cannot be determined. Thus,

The value of, $\sqrt{16} + \sqrt{9}$, is regarded as 7, although -7, $+1$, and -1 are other possible values.

When an even root of an equation is extracted, the plus and minus signs are both used with one member of the resulting equation. Thus,

If $$x^2 + 2x + 1 = 4$$
$$x + 1 = \pm 2$$

When the sign of an even root can be determined, that sign may always be used. Illustrations occur in the solution of irrational equations.

1. Find the value of x:

$$x + \sqrt{x} = 2$$
$$x + x^{\frac{1}{2}} + \tfrac{1}{4} = 2\tfrac{1}{4}$$
$$x^{\frac{1}{2}} + \tfrac{1}{2} = \pm 1\tfrac{1}{2}$$
$$x^{\frac{1}{2}} = 1, \text{ or } -2$$
$$x = 1, \text{ or } 4$$

If $x = 1$, the sign of $x^{\frac{1}{2}}$ is '+'; if $x = 4$, the sign of $x^{\frac{1}{2}}$ is '−'. Substituting the first value of x and the corresponding value of $x^{\frac{1}{2}}$, $1 + 1 = 2$. Substituting the second value of x and the corresponding value of $x^{\frac{1}{2}}$, $4-2=2$.

2. Find the value of x:

$$x - \sqrt{25 - x^2} = 7$$
$$x - 7 = \sqrt{25 - x^2}$$
$$x = 4,\ 3$$

If $x = 4$, the sign of $\sqrt{25 - x^2}$ is '−' because $x - 7 = 4 - 7$, or -3. If $x = 3$, the sign of $\sqrt{25 - x^2}$ is '−' for the same reason.

3. Find the value of x:

$$x - \sqrt{25 - x^2} = 1$$
$$x - 1 = \sqrt{25 - x^2}$$
$$x = 4,\ -3$$

If $x = 4$, the sign of $\sqrt{25 - x^2}$ is '+' because $x - 1 = 4 - 1$, or 3. If $x = -3$, the sign of $\sqrt{25 - x^2}$ is '−' because $x - 1 = -3 - 1$, or -4.

NOTE. The solution of Ex. 2 is commonly stated as impossible because neither 4 nor 3 will satisfy the equation. The solution of Ex. 3 is commonly said to yield the extraneous root, -3, because -3 will not satisfy the equation. This is on the assumption that $\sqrt{25 - x^2}$ must be '+'.

TERMS SIMPLIFIED

According to the convention of p. 246, if a factor is removed from a radical of an even index, it is regarded when outside of the radical as having the '+' sign only. Thus,

$$\sqrt{18} = \sqrt{9 \times 2} = +3\sqrt{2}$$

$$\sqrt{-18} = \sqrt{9 \times 2 \times -1} = +3\sqrt{2}\sqrt{-1}$$

An imaginary is not in its simplest form until the factor containing the even root of the negative number is $\sqrt[n]{-1}$.

Simplify:

4. $\sqrt{-9}$
5. $\sqrt{-18}$
6. $\sqrt{-32}$
7. $\sqrt{-12}$
8. $\sqrt{-10}$
9. $3\sqrt{-54}$
10. $-\sqrt{-24}$
11. $-3\sqrt{-50}$

Ex. 4. $\sqrt{-9} = \sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1} = 3\sqrt{-1}$.

Ex. 10. $-\sqrt{-24} = -\sqrt{4 \times 6 \times -1} = -2\sqrt{6}\sqrt{-1}$.

It follows from the convention of p. 246, that if a '$-$' sign is outside of a radical of an even index, it cannot be placed under the radical, but must remain outside. Thus,

$$-3\sqrt{2} = -\sqrt{9 \times 2} = -\sqrt{18}$$

$$-3\sqrt{-2} = -\sqrt{9 \times -2} = -\sqrt{-18}$$

Introduce under the radical:

12. $-2\sqrt{-24}$
13. $-3\sqrt{-4}$
14. $-5\sqrt{-3}$
15. $6\sqrt{-2}$
16. $3\sqrt{-5}$
17. $2\sqrt{-6}$

Ex. 12. $-2\sqrt{-24} = -\sqrt{4 \times -24} = -\sqrt{-96}$.

NOTE. The pupil should understand clearly that the above agreement has been reached for the sake of simplicity and that it should not be violated.

ADDITION AND SUBTRACTION

The preparation for addition and subtraction consists in the reduction of each term to its simplest form.

Add:

18. $\sqrt{-27}$, $\sqrt{-48}$, $\sqrt{-75}$

19. $\sqrt{-9}$, $\sqrt{-4}$, $\sqrt{-16}$

20. $\sqrt{-50}$, $-\sqrt{-32}$, $-\sqrt{-8}$

21. $3\sqrt{-72}$, $2\sqrt{-98}$, $\sqrt{-50}$

Subtract:

22. $\sqrt{-8}$ from $\sqrt{-18}$

23. $\sqrt{-12}$ from $\sqrt{-75}$

24. $\sqrt{-20}$ from $\sqrt{-45}$

25. $\sqrt{-28}$ from $\sqrt{-63}$

Ex. 18.
$$\begin{aligned}\sqrt{-27} &= 3\sqrt{3}\sqrt{-1}\\ \sqrt{-48} &= 4\sqrt{3}\sqrt{-1}\\ \sqrt{-75} &= \underline{5\sqrt{3}\sqrt{-1}}\\ & \quad 12\sqrt{3}\sqrt{-1}\end{aligned}$$

Ex. 22.
$$\begin{aligned}\sqrt{-18} &= 3\sqrt{2}\sqrt{-1}\\ \sqrt{-8} &= \underline{2\sqrt{2}\sqrt{-1}}\\ & \quad \sqrt{2}\sqrt{-1}\end{aligned}$$

NOTE. These operations could be performed by the reduction of each term to a form containing a common imaginary factor, but it is better to simplify. Thus, $\sqrt{-18}-\sqrt{-8}=3\sqrt{-2}-2\sqrt{-2}=\sqrt{-2}$.

MULTIPLICATION AND DIVISION

The preparation for multiplication and division consists in the reduction of each term to its simplest form. It is then possible to avoid the use of the case of *exponents the same.*

Multiply:

26. $\sqrt{-4}$ by $\sqrt{-9}$

27. $\sqrt{-18}$ by $\sqrt{12}$

28. $\sqrt{-72}$ by $-\sqrt{-2}$

29. $\sqrt{-2}$ by $\sqrt{-24}$

30. $-3\sqrt{27}$ by $2\sqrt{-2}$

31. $-\sqrt{-5}$ by $-3\sqrt{-10}$

32. $2\sqrt{-32}$ by $4\sqrt{2}$

33. $3\sqrt{-15}$ by $-2\sqrt{-5}$

Ex. 26. $\sqrt{-4}\times\sqrt{-9}=2\sqrt{-1}\times 3\sqrt{-1}=6\times(\sqrt{-1})^2=6\times -1=-6$.
$\sqrt{-1}\times\sqrt{-1}$ is the case when the bases are the same; the result is $(-1)^{\frac{1}{2}+\frac{1}{2}}$, or -1.

MULTIPLICATION AND DIVISION

Before division by the irrational factor is performed, it is necessary to make the divisor rational.

Divide:

34. -6 by $\sqrt{-4}$

35. $\sqrt{-216}$ by $\sqrt{12}$

36. 12 by $-\sqrt{-2}$

37. $-4\sqrt{3}$ by $\sqrt{-2}$

38. $-18\sqrt{-6}$ by $-3\sqrt{27}$

39. $-15\sqrt{2}$ by $-\sqrt{-5}$

40. $64\sqrt{-1}$ by $2\sqrt{-32}$

41. $30\sqrt{3}$ by $3\sqrt{-15}$

Ex. 34. $-6 \div \sqrt{-4} = \dfrac{-6}{2\sqrt{-1}} = \dfrac{-3}{\sqrt{-1}} = \dfrac{-3\sqrt{-1}}{-1} = 3\sqrt{-1} = \sqrt{-9}.$

Ex. 35. $\sqrt{-216} \div \sqrt{12} = \dfrac{6\sqrt{6}\,\sqrt{-1}}{2\sqrt{3}} = 3\sqrt{2}\,\sqrt{-1} = 3\sqrt{-2} = \sqrt{-18}.$

Ex. 36. $12 \div -\sqrt{-2} = \dfrac{12}{-\sqrt{2}\,\sqrt{-1}} = \dfrac{12\sqrt{2}\,\sqrt{-1}}{2} = 6\sqrt{2}\,\sqrt{-1} = \sqrt{-72}.$

Note. In every case, we multiply or divide both dividend and divisor by the same number. In Ex. 34, we divide both terms by 2, then multiply both terms by $\sqrt{-1}$, and so on.

INVOLUTION AND EVOLUTION

As a preparation, each term must be reduced to its simplest form.

Raise $\sqrt{-1}$ *to:*

42. The 2d power

43. The 3d power

44. The 4th power

45. The 5th power

46. The 6th power

47. The 7th power

Ex. 43. $(\sqrt{-1})^3 = (-1)^{\frac{3}{2}} = (-1)^{1+\frac{1}{2}} = -1\sqrt{-1} = -\sqrt{-1}.$

Ex. 44. $(\sqrt{-1})^4 = (-1)^{\frac{4}{2}} = (-1)^2 = +1.$

Ex. 47. $(\sqrt{-1})^7 = (-1)^{\frac{7}{2}} = (-1)^{3+\frac{1}{2}} = -1\sqrt{-1} = -\sqrt{-1}.$

Note. $(\sqrt{-1})^3 = (-1)^{\frac{3}{2}} = \sqrt{(-1)^3} = \sqrt{-1}$, but this cannot be allowed by the convention of p. 246. $(\sqrt{-1})^2 = -1$; $-1 \times \sqrt{-1} = -\sqrt{-1}$ because the '$-$' sign cannot be introduced under the radical.

48. There must be three cube roots of 8 because the equation, $x^3-8=0$, has three roots. Find them.

If $x^3-8=0$, $(x-2)(x^2+2x+4)=0$; hence, $x-2=0$, and $x^2+2x+4=0$; hence, $x=2$, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

49. By the binomial theorem, prove that $-1+\sqrt{-3}$ is a cube root of 8.

$$-1+\sqrt{-3}=-1+\sqrt{3}\sqrt{-1}$$

$$\begin{aligned}(-1+\sqrt{3}\sqrt{-1})^3 &= (-1)^3+3(-1)^2\sqrt{3}\sqrt{-1}+3(-1)(\sqrt{3})^2(\sqrt{-1})^2\\ &\quad +(\sqrt{3})^3(\sqrt{-1})^3\\ &= -1+3\times 1\times\sqrt{3}\sqrt{-1}+3\times -1\times 3\times -1\\ &\quad +3\sqrt{3}\times -\sqrt{-1}\\ &= -1+3\sqrt{3}\sqrt{-1}+9-3\sqrt{3}\sqrt{-1}\\ &= 8\end{aligned}$$

50. By multiplication, prove that $-1+\sqrt{-3}$ is a cube root of 8.

$$\begin{array}{l} -1+\sqrt{3}\sqrt{-1}\\ \underline{-1+\sqrt{3}\sqrt{-1}}\\ 1-\sqrt{3}\sqrt{-1}\\ \underline{\quad -\sqrt{3}\sqrt{-1}+3\times -1}\\ 1-2\sqrt{3}\sqrt{-1}-3\\ \quad -2-2\sqrt{3}\sqrt{-1},\ \text{2d power}\end{array}\qquad\qquad \begin{array}{l} -2-2\sqrt{3}\sqrt{-1}\\ \underline{-1+\ \ \sqrt{3}\sqrt{-1}}\\ 2+2\sqrt{3}\sqrt{-1}\\ \underline{\quad -2\sqrt{3}\sqrt{-1}-2\times 3\times -1}\\ 2\qquad\qquad\qquad +6\\ \qquad 8,\ \text{3d power}\end{array}$$

51. By the binomial theorem, prove that $(-1-\sqrt{-3})^3=8$.

52. By multiplication, prove that $(-1-\sqrt{-3})^3=8$.

53. How many values has the 6th root of 64?

Six, because $x^6-64=0$ is an equation of the 6th degree.

Numerical expressions may be exactly represented by lines even when they cannot be expressed as integers and fractions. Thus, $\sqrt{2}$ may be represented by the diagonal of a square whose base is 1. There are devices also for representing imaginaries by lines, but their discussion does not fall within the limits of this treatise.

THE BINOMIAL THEOREM

NEGATIVE AND FRACTIONAL EXPONENTS

A rigorous proof of the binomial theorem is not attempted in this book, but in more advanced treatises the laws stated on p. 55 are demonstrated for positive, for negative, and for fractional exponents. The pupil may apply these laws for the sake of practice in the various operations, and for the sake of a comprehensive view of the scope of the binomial theorem.

1. (*a*) Expand $(a+b)^{-1}$ by the binomial theorem.

$$(a+b)^{-1} = a^{-1} - a^{-2}b + a^{-3}b^2 - a^{-4}b^3 + a^{-5}b^4 - \cdots$$

The exponents of a are $-1, -2, -3, -4, \cdots$; of b, $0, 1, 2, 3, 4, \cdots$.

The coefficient of the 1st term is 1; of the 2d, -1; of the 3d, $\frac{-1 \times -2}{2}$, or 1; of the 4th, $\frac{1 \times -3}{3}$, or -1; $\cdots$.

(*b*) Expand $(a+b)^{-1}$ by division.

$$(a+b)^{-1} = \frac{1}{a+b} \quad \text{Prop. VII}$$

$$\frac{1}{a+b} = \frac{a^0}{a+b} \quad \text{Prop. VIII}$$

$$\begin{array}{r|l} a+b)\,a^0 & (a^{-1} - a^{-2}b + a^{-3}b^2 - \cdots \\ \underline{a^0 + a^{-1}b} & \\ -a^{-1}b & \\ \underline{-a^{-1}b - a^{-2}b^2} & \\ a^{-2}b^2 & \end{array}$$

2. (*a*) Expand $(a+b)^{\frac{1}{2}}$ by the binomial theorem.

$$(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}a^{-\frac{5}{2}}b^3 - \cdots$$

The exponents of a are $\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$; of b, $0, 1, 2, 3, 4, \cdots$.

The coefficient of the 1st term is 1; of the 2d, $\frac{1}{2}$; of the 3d, $\frac{1}{2} \times -\frac{1}{2} \times \frac{1}{2}$, or $-\frac{1}{8}$; of the 4th, $-\frac{1}{8} \times -\frac{3}{2} \times \frac{1}{3}$, or $+\frac{1}{16}$; $\cdots$.

(*b*) Expand $(a+b)^{\frac{1}{2}}$ by extracting the square root.

$$(x+y)^2 = x^2 + 2\,xy + y^2$$
$$= x^2 + (2\,x + y)y$$

$$\begin{array}{r|l}
a+b & \underline{a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b - \frac{1}{8}a^{-\frac{3}{2}}b^2 + \cdots} \\
 & a \\
2\,a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b & +b \\
 & \underline{+b + \frac{1}{4}a^{-1}b^2} \\
2\,a^{\frac{1}{2}} + a^{-\frac{1}{2}}b - \frac{1}{8}a^{-\frac{3}{2}}b^2 & -\frac{1}{4}a^{-1}b^2 \\
 & \underline{-\frac{1}{4}a^{-1}b^2 - \frac{1}{8}a^{-2}b^3 + \frac{1}{64}a^{-3}b^4}
\end{array}
\qquad
\begin{array}{l}
x = a^{\frac{1}{2}} \\
y = \frac{1}{2}a^{-\frac{1}{2}}b \\
\hline
x = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b \\
y = -\frac{1}{8}a^{-\frac{3}{2}}b^2
\end{array}$$

3. Expand $(a-b)^{-2}$ by the binomial theorem.

$$(a+b)^{-2} = a^{-2} - 2\,a^{-3}b + 3\,a^{-4}b^2 - 4\,a^{-5}b^3 + \cdots$$
$$(a-b)^{-2} = a^{-2} + 2\,a^{-3}b + 3\,a^{-4}b^2 + 4\,a^{-5}b^3 + \cdots$$

We find the value of $(a+b)^{-2}$ and then, beginning with the second term, we change the sign of every alternate term.

4. Expand $(a-b)^{-m}$ by the binomial theorem.

$$(a-b)^{-m} = a^{-m} + ma^{-(m+1)}\,b + \frac{m(m+1)}{\lfloor 2}\,a^{-(m+2)}\,b^2$$
$$+ \frac{m(m+1)(m+2)}{\lfloor 3}\,a^{-(m+3)}\,b^3 + \cdots$$

NOTE. $\lfloor 2 = 1 \times 2$; $\lfloor 3 = 1 \times 2 \times 3$; see p. 29. $-m-1 = -(m+1)$; $-m-2 = -(m+2)$; ….

Expand by the binomial theorem:

5. $(a+b)^{-\frac{1}{3}}$

6. $(a-b)^{\frac{1}{2}}$

7. $(a+b)^{\frac{2}{3}}$

8. $(a-b)^{-\frac{3}{4}}$

INEQUALITIES

DISCUSSION

Affecting both members of an equation in the same manner does not disturb the sign of equality. Thus,

The equality is not disturbed: when both members are increased, diminished, multiplied, or divided, by the same number; when both members are raised to the same power, or depressed to the same root; and when the corresponding members of two equations are added, subtracted, multiplied, or divided.

Affecting both members of an inequality in the same manner does not disturb the sign of inequality in some cases, but reverses it in others. Thus,

Subtracting an equation from an inequality member from member does not disturb the sign of inequality, but subtracting an inequality from an equation member from member reverses the sign of inequality.

$$\begin{array}{lcccc} & 8>6 & 6<8 & 10=10 & 10=10 \\ & 10=10 & 10=10 & 8>6 & 6<8 \\ \hline \text{Subtracting,} & -2>-4 & -4<-2 & 2<4 & 4>2 \end{array}$$

In advanced treatises on algebra, the different cases in inequalities are developed and proved. In this book, the pupil is advised to work with inequalities after the general method of working with equations, but to test each step before taking it, in order to ascertain whether the proposed operation will reverse the sign.

1. If 4 is added to a number, the sum is greater than 10. Find the number.

Let $x =$ the number. $\qquad x + 4 > 10 \qquad (1)$

$$x > 6 \qquad (2)$$

If (1) were an equation, we should transpose and unite. Before taking this step, we must determine whether transposing a term will reverse the sign. We may take an inequality, as $10 > 6$, and try the effect of subtracting from each member a number, as 2, that will leave each remainder positive, $8 > 4$; a number, as 8, that will leave one remainder negative, $2 > -2$; a number, as 11, that will leave both remainders negative, $-1 > -5$. Having discovered that a term may be transposed without disturbing the sign, we transpose in (1) and obtain (2).

NOTE. Since the value of x may approach but can never reach 6, 6 is said to be the limit of the value of x.

2. If 10 is diminished by 3 times a number, the result is less than 4. Find the number.

Let $x =$ the number.

$10 - 3x < 4$ (1) $\qquad -3x < -6$ (2) $\qquad x > 2$ (3)

We obtain (2) by transposing in (1). If (2) were an equation, we should divide by -3. Before doing so, we make various tests and discover that dividing by a negative number reverses the sign.

3. Two times a number plus 3 times a second is greater than 13; 5 times the first minus 2 times the second equals 4. Find the numbers.

Let $x =$ 1st; $y =$ 2d.

$$\begin{array}{lll} 2x + 3\underline{y} > 13 & 10x + 15y > 65 & 19y > 57 \\ 5x - 2\underline{y} = 4 & 10x - 4\underline{y} = 8 & y > 3;\ x > 2 \end{array}$$

NOTE. The pupil should satisfy himself that each step is admissible before he takes it.

4. If 4 times a number is subtracted from its square, the result is greater than 5. Find the number.

Let $x =$ the number.

$$\begin{array}{lll} x^2 - 4x > 5 & (1) & x - 2 > 3, \text{ or } x - 2 < -3 \\ x^2 - 4x + 4 > 9 & (2) & x > 5, \text{ or } x < -1 \end{array}$$

Here, x has two limits, 5 and -1; any number greater than 5 and any number less than -1 will satisfy the conditions.

NOTE. If $y^2 > 9$, $y > 3$ or < -3; hence the square root of (2) is $x - 2 > 3$, or $x - 2 < -3$.

INFINITY AND INFINITESIMALS

DEVELOPMENT

A number may be finite, infinite, or infinitesimal. A number however large or small is finite when it is not greater and not less than an assignable number. A number is infinite when it is greater than any assignable number; a number is infinitesimal when it is less than any assignable number.

Thus, 1 with a sextillion ciphers, and a fraction whose numerator is 1 and whose denominator is 1 with a sextillion ciphers, are finite numbers because larger and smaller numbers may be named. The number of drops in the ocean is finite because a larger number may be named. The number of seconds in eternity is infinite; the fraction whose numerator is 1 and whose denominator is the number of seconds in eternity is infinitesimal.

Infinity, an infinite number, is represented by ∞; an infinitesimal, by 0.

The symbol, ∞, does not always represent the same number, but may have an infinite number of different values. In the same way, the symbol, 0, may represent any one of an infinite number of infinitesimals.

Thus, the number of cubic inches in space is ∞, the number of cubic feet in space is ∞, the number of cubic yards in space is ∞. It would be impossible to assign a separate symbol for every infinity and for every infinitesimal.

The symbol, 0, is thus used for absolutely nothing and for an infinitesimal. Some confusion results, but no suggestion for separate characters has been generally adopted.

An infinitesimal has absolute zero for its limit; infinity has no limit.

Thus, if $\frac{1}{2}$ is divided by 2, the result by 2, the last result by 2, and so on for an infinite number of times, the value of the fraction approaches, but can never become, absolute zero. Hence, absolute zero is the limit of the infinitesimal. The number of seconds in eternity has no limit; infinity has no limit.

Infinity divided by infinity, $\infty \div \infty$, produces any number, finite, infinite, or infinitesimal.

Thus, if the dividend is the number of seconds in eternity and the divisor is the number of minutes, $\infty \div \infty = 60$. If the dividend is the number of infinitesimals of time in eternity and the divisor is the number of seconds, $\infty \div \infty = \infty$. If the dividend is the number of seconds in eternity and the divisor the number of infinitesimals of time, $\infty \div \infty = 0$ (an infinitesimal).

An infinitesimal divided by an infinitesimal, $0 \div 0$, produces any number, finite, infinite, or infinitesimal.

Thus, if the infinitesimal whose numerator is 1 and whose denominator is the number of seconds in eternity, is divided by the infinitesimal whose numerator is 1 and whose denominator is the number of minutes in eternity, $0 \div 0 = \frac{1}{60}$. As in the preceding paragraph, it may be shown that $0 \div 0 = 60$, $0 \div 0 = \infty$, and $0 \div 0 = 0$.

A finite number divided by infinity, $a \div \infty$, produces an infinitesimal; a finite number divided by an infinitesimal, $a \div 0$, produces infinity. The pupil should prove these relations.

If 0 is an infinitesimal, find the value of:

1. 0×0

2. 0×6

3. $\dfrac{0}{6}$

4. $\dfrac{6}{0}$

5. 6×0

6. $\infty \times \infty$

7. $\infty \times 6$

8. $\dfrac{6}{\infty}$

9. $\dfrac{\infty}{6}$

10. $\infty \times 6$

11. $0 \times \infty$

12. $\infty \times 0$

13. $\dfrac{0}{0}$

14. $\dfrac{\infty}{\infty}$

15. $0 \div \infty$

DEVELOPMENT

In geometrical progression, the case of finding the sum of an infinite series when $r < 1$, involves infinitesimals and infinity.

16. Find the sum of an infinite number of terms of the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

$$\left.\begin{array}{l} a = 1 \\ r = \frac{1}{2} \\ l = 0 \\ n = \infty \end{array}\right| s = ? \qquad s = \frac{rl - a}{r - 1} = \frac{0 - 1}{-\frac{1}{2}} = 2 - 0$$

The sum is 2 minus an infinitesimal. The sum can never be exactly 2 because an infinitesimal can never become absolutely nothing. Sometimes, the sum is said to be 2; the correct expression is, the limit of the sum is 2; *i.e.*, the sum is constantly approaching 2 and comes to differ from 2 by less than any assignable value.

Find the limit of the sum of an infinite number of terms:

17. $1, \frac{1}{3}, \frac{1}{9}, \cdots$

18. $2, \frac{1}{2}, \frac{1}{4}, \cdots$

19. $36, 24, 16, \cdots$

20. $64, 8, 1, \cdots$

The limit of the value of a repetend may also be found.

21. Find the limit of the value of $.\dot{5}$, or $.555 \cdots$.

$.555 \cdots = .5 + .05 + .005 + .0005 + \cdots$. $\therefore a = .5,\ r = .1,\ l = 0,\ n = \infty$;

$s = \frac{0 - .5}{-.9} = \frac{.5 - 0}{.9} = \frac{5}{9} - 0$; $.\dot{5} = \frac{5}{9}$ minus an infinitesimal; the limit of the value is $\frac{5}{9}$. It is customary to say that $\frac{5}{9} = .5555 + \cdots$, but the '+' must be understood as meaning an infinite number of 5's together with an infinitesimal.

22. Find the limit of the value of $.\dot{6}3\dot{4}$, or $.634634634 \cdots$.

23. (*a*) If an insect at the end of a board 20 feet long could jump half the length of the board, then half the distance that is left, then half of the remaining distance, and so on, how many jumps would be required to cover a distance of 19.99 feet?

(*b*) What is the limit of the sum of an infinite number of such jumps?

PROPOSITIONS REVIEWED

The first part of this book closes with indeterminate equations, p. 126, but the propositions of this part are applicable to examples discussed farther on. In order that pupils may review the first part before beginning the second, and that the logical treatment may be preserved, up to Prop. L, all examples belonging to the first part are printed in large type, and all examples belonging to the second part in small type.

I

1. The area of a triangle is equal to $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the half sum of its sides and a, b, and c are its sides. Express without the use of letters.

2. In percentage, the base multiplied by the rate is equal to the percentage. Express this truth by means of letters.

II

1. If by agreement, $+3a$ means that a is used 3 times as an addend, what must $-3a$ mean?

2. If by agreement a^{+3} means that a is used 3 times as a multiplier, what must a^{-3} mean?

III–V

1. Why does $+3a$ mean $a+a+a$, or that a is used 3 times as an addend?

2. Why does $-3a$ mean $-a-a-a$, or that a is used 3 times as a subtrahend?

3. Why does $0a$ mean that a is used the same number of times both as an addend and as a subtrahend?

VI–VIII

1. Why does a^{+3} mean $a \times a \times a$, or that a is used 3 times as a multiplier?

2. Why does a^{-3} mean $\frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$, or that a is used 3 times as a divisor?

3. Why does a^0 mean that a is used the same number of times both as a multiplier and as a divisor?

IX–X

1. Add: $+6, +8$; $-3, -4$; $+8, -6$; $-8, +6$.
2. Add: $-ax, +bx, +cx$; $(a-b)x, (a+b)x, (a-3b)x$.
3. Add: $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}, \frac{a^2-2ab+b^2}{a^2+2ab+b^2}$.
4. Add: $\sqrt{27}, \sqrt{75}, \sqrt{48}$; $\sqrt{\frac{1}{2}}, -3\sqrt{\frac{2}{9}}, \sqrt{\frac{1}{18}}$; $\sqrt{-24}, -\sqrt{-6}, -\sqrt{-54}$.
5. Add: $\infty, 6$; $\infty, 0$; ∞, ∞.

XI–XII

1. Subtract: 8 from 6; -8 from -6; 2 from -6; 6 from -2.
2. Subtract: bx from ax; $-dx$ from cx.
3. Subtract: $(a^2-2ab+b^2)x$ from $(a^2+2ab+b^2)x$.
4. Subtract: $\frac{a-b}{a+b}$ from $\frac{a+b}{a-b}$.
5. Subtract: $\sqrt{50}$ from $\sqrt{72}$; $-\sqrt{20}$ from $\sqrt{45}$.
6. Subtract: $\sqrt{\frac{1}{5}}$ from $\sqrt{\frac{9}{20}}$.
7. Subtract: $\sqrt{-8}$ from $\sqrt{-18}$; $-\sqrt{-12}$ from $-\sqrt{-27}$.
8. Subtract 6 from ∞; 0 from ∞; ∞ from ∞.

XIII–XVII

1. Multiply: $+6$ by $+8$; -6 by -8; -6 by $+8$; $+8$ by -6.

2. State the case in multiplication: $5^2 \times 5^3$; $4^3 \times 6^3$; $4^2 \times 8^3$; $2^5 \times 2^5$.

3. Multiply: 5^2 by 5^3; 4^3 by 6^3; 5^a by 5^b; 6^a by 3^a.

4. Multiply: 2^3 by 4^2; 3^4 by 9^2; 8^2 by 2^3; $2^5 \times 2^5$. Reduce to the case when the bases are the same.

5. In Ex. 4, reduce to the case when the exponents are the same, and multiply again.

6. Multiply: 2^3 by 3^2; 4^3 by 5^2; 9^2 by 4. Can you readily reduce these to the case when the bases are the same?

7. Multiply: $3x^2 - 2xy + y^2$ by $2x^2 - 3xy - 6y^2$.

8. State the case in multiplication: $\sqrt{2} \times \sqrt[3]{2}$; $\sqrt[3]{12} \times \sqrt[3]{9}$; $\sqrt{3} \times \sqrt[3]{2}$; $\sqrt[3]{4} \times \sqrt[3]{4}$.

9. Place the factor under the radical: $3\sqrt{6}$; $2\sqrt[3]{4}$; $2\sqrt[4]{5}$. What case in multiplication is this? How do we proceed?

10. Place the factor under the radical: $-2\sqrt{3}$. Why is the answer regarded as $-\sqrt{12}$ and not as $+\sqrt{12}$?

11. Multiply: $\sqrt{12}$ by $3\sqrt{6}$; $-\sqrt{8}$ by $-2\sqrt{2}$; $\sqrt{6} - 3$ by $\sqrt{6} - 2$.

12. Place the factor under the radical: $3\sqrt{-1}$; $-3\sqrt{-1}$.

13. Multiply: $\sqrt{-4}$ by $\sqrt{-9}$. Is the answer ever regarded as $\sqrt{36}$? Why not?

14. Multiply: ∞ by 6; ∞ by ∞; ∞ by 0.

XVIII–XXII

1. Divide: 16 by -2; -8 by -2; -18 by 6.

2. State the case in division: $8^6 \div 8^2$; $8^3 \div 2^3$; $4^3 \div 3^2$; $4^3 \div 4^3$.

3. Divide: 8^6 by 8^2; 8^3 by 2^3; 8^a by 8^b; 8^a by 2^a.

4. Divide: 2^8 by 4^2; 27^2 by 9^3; 8^4 by 4^3; 4^3 by 4^3. Reduce to the case when the bases are the same.

5. In Ex. 4, reduce to the case when the exponents are the same, and divide again.

6. Divide: 6^3 by 5^2; 9^2 by 2^4; 5^3 by 2^4. Can you readily reduce these to the case when the bases are the same?

7. Divide: $9x^4 + 2x^2y^2 + y^4$ by $3x^2 - 2xy + y^2$.

8. State the case in division: $\sqrt{2} \div \sqrt[6]{2}$; $\sqrt[3]{12} \div \sqrt[3]{4}$; $\sqrt{3} \div \sqrt[3]{2}$; $\sqrt{2} \div \sqrt{2}$.

9. Remove the factor from under the radical: $\sqrt{54}$; $\sqrt[3]{32}$; $\sqrt[4]{80}$. What case in division is this? How do we proceed?

10. Remove the factor from under the radical: $-\sqrt{12}$. Why is the answer regarded as $-2\sqrt{3}$ and not as $+2\sqrt{3}$?

11. Divide: $\sqrt{72}$ by $-\sqrt{2}$; $-\sqrt{8}$ by $-\sqrt{2}$; $12 - 5\sqrt{6}$ by $\sqrt{6} - 3$.

12. Remove the factor from under the radical: $\sqrt{-9}$; $-\sqrt{-9}$.

13. Divide: $\sqrt{36}$ by $\sqrt{-9}$. Is the answer ever regarded as $\sqrt{-4}$? Why not?

14. Divide: ∞ by 6; 6 by ∞; ∞ by ∞; ∞ by 0; 0 by ∞.

XXIII

1. Declare the square of: $a - b + c - d$.

2. Declare the square of: $(x + y) - (x - y)$.

3. Factor: $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$.

4. Declare the square of: $3 - \sqrt{2} + \sqrt{3}$.

5. Declare the factors: $5 - \sqrt{24}$.

6. Declare the square of: $\sqrt{-3} - \sqrt{-2}$.

7. Declare the factors: $-5 + 2\sqrt{6}$.

XXIV

1. Declare the product of: $4ab + 3cd$ and $4ab - 3cd$.

2. Declare the product of: $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

3. Declare the product of: $a + b - c$ and $a - b + c$.

4. Declare the product of: $(x - y)(x + y)(x^2 + y^2)(x^4 + y^4)$.

5. Declare the factors of: $9x^2y^2 - 4z^2$.

6. Declare the factors of: $a^4 + a^2b^2 + b^4$.

7. Declare the factors of: $a^2 - b^2 - c^2 - 2bc$.

8. Declare the factors of: $4x^2y^2 - (x^2 + y^2 - z^2)^2$.

9. Declare the factors of: $a^8 - b^8$.

10. Declare the product of: $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$.

11. Declare the product of: $\sqrt{-3} + \sqrt{-2}$ and $\sqrt{-3} - \sqrt{-2}$.

XXV

1. Declare the product of: $a-6$ and $a+5$.
2. Declare the product of: $3xy-b$ and $3xy+2b$.
3. Declare the factors of: x^2-5x+6.
4. Declare the factors of: $(a-b)^2-5(a-b)+6$.

XXVI

1. Declare the product: $(x+y)^5$; $(x-y)^5$.
2. Declare the product: $(2x+3y)^4$; $(2x-3y)^4$.
3. Declare the product: $(x+y)^m$; $(x-y)^m$.
4. Declare the product: $(x+y)^{-1}$; $(x-y)^{-1}$.
5. Declare the product: $(x+y)^{-\frac{1}{2}}$; $(x-y)^{-\frac{1}{2}}$.

XXVII

1. Declare the quotient of: $x^{12}-y^{12}$ by $x+y$, by $x-y$.
2. Declare the quotient of: $x^{12}-y^{12}$ by x^2+y^2, by x^2-y^2.
3. Declare the quotient of: $x^{12}-y^{12}$ by x^3+y^3, by x^3-y^3.
4. Declare the quotient of: $x^{12}-y^{12}$ by x^4+y^4, by x^4-y^4.
5. Declare the quotient of: $x^{12}-y^{12}$ by x^6+y^6, by x^6-y^6.
6. Declare the quotient of: $x^{12}+y^{12}$ by x^4+y^4, by x^3+y^3.
7. Declare the quotient of: x^n+y^n by $x+y$, by $x-y$.
8. Declare the quotient of: x^n-y^n by $x+y$, by $x-y$.

XXVIII-XXXII

1. Write the formula for any number in the decimal system.
2. State which of the following numbers are exactly divisible by 2: 36874; 96285; 7382.
3. State which of the following numbers are exactly divisible by 3: 68454; 7283; 4386; 576.

4. State which of the following numbers are exactly divisible by 4: 6276; 3214; 6284; 9326.

5. State which of the following numbers are exactly divisible by 11: 3462; 3784; 235; 682.

6. Prove that the remainder found by subtracting the sum of its digits from a number is exactly divisible by 9.

XXXIII–XXXVI

1. Find the H. C. F.: $36x^2y^2$, $16xyz$, $32x^3y^3z^3$.

2. Find the H. C. F.: $6(x^2-y^2)$, $8(x-y)^2$, $4(x^2+xy-2y^2)$.

3. Find two expressions which have the same H. C. F. as: $2x^2-11x+15$ and $3x^2-5x-12$.

4. Find lower expressions which have the same H. C. F. as: $2x^2-13x+15$ and $4x^2-16x+15$.

5. Find the H. C. F.: $11x^4-9ax^3-a^2x^2-a^4$ and $13x^4-10ax^3-2a^2x^2-a^4$.

XXXVII

1. Find the L. C. M.: $18x^2y$, $32xy^2$, $27x^3y^3$.

2. Find the L. C. M.: x^2-y^2, $x^2+2xy+y^2$, $x^2-2xy+y^2$.

3. Find the L. C. M.: $4x^2-5xy+y^2$, $3x^3-3x^2y+xy^2-y^3$.

XXXVIII–XLV

1. Reduce to lowest terms: $\dfrac{75x^2y^3}{125x^3y}$; $\dfrac{2x^3+3x^2y-y^3}{4x^3+xy^2-y^3}$.

2. Reduce to equivalent fractions having the L. C. D.:

$$\frac{x-3}{x-5},\ \frac{x+2}{x-5},\ \frac{x-1}{x^2-25}.$$

3. Reduce to a whole or a mixed number: $\dfrac{x^3-4x^2+x+4}{x-3}$.

4. Reduce to a fraction: $x-5-\dfrac{x-2}{x-3}$.

5. Add: $\dfrac{1}{x-2}$, $\dfrac{1}{x+3}$, $\dfrac{1}{x^2+x-6}$.

6. Subtract: $\dfrac{x-3}{x-4}$ from $\dfrac{x-4}{x-3}$.

7. Multiply: $\dfrac{x^2-5x+6}{x^2-x-2}$ by $\dfrac{x^2-1}{x^2-9}$.

8. Divide: $\dfrac{a^3+b^3}{a^3-b^3}$ by $\dfrac{a^2-ax+x^2}{a^2+ax+x^2}$.

9. Simplify: $\dfrac{x-3+\dfrac{x-1}{x-2}}{x-5-\dfrac{x+1}{x-2}}$; $\dfrac{\dfrac{1}{x-a}-\dfrac{1}{x+a}}{\dfrac{1}{x^2-a^2}-\dfrac{1}{x^2+a^2}}$.

XLVI–XLIX

1. Preparatory to finding the value of x, transpose and unite:

$$ax-b=-a+bx.$$

2. Find the value of x: $\dfrac{x-3}{3}-\dfrac{x+2}{4}=3$.

3. Find the value of x: $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}$.

4. Find the value of x: $\dfrac{x^2-2x+3}{x-2}-\dfrac{x^2+3x-4}{x+3}=\dfrac{29}{x^2+x-6}$.

5. Find the values of x and y by addition and subtraction:

$$\begin{cases} 3x-2y=11 \\ 5x-3y=19 \end{cases}$$

6. In Ex. 5, find the values of x and y by comparison.

7. In Ex. 5, find the values of x and y by substitution.

8. Find the values of x and y: $\dfrac{3}{x}-\dfrac{4}{y}=1$ and $\dfrac{5}{x}+\dfrac{2}{y}=6$.

9. Find the values of x, y, and z: $\begin{cases} 2x - 3y + 3z = 6 \\ 3x - y + 2z = 7 \\ 6x - y - z = 1 \end{cases}$

10. Find the values of x and y in positive integers: $2x - 3y = 6$.

L–LIII

1. Raise: a^3 to the 2d power; a^3 to the -2d power; a^3 to the $\frac{1}{3}$ power; a^3 to the $-\frac{1}{3}$ power; a^3 to the $\frac{2}{3}$ power; a^3 to the $-\frac{2}{3}$ power.

2. Depress: a^6 to the 2d root; a^{-6} to the -2d root; a to the $\frac{1}{3}$ root; a^{-1} to the $-\frac{1}{3}$ root; a^2 to the $\frac{2}{3}$ root; a^{-2} to the $-\frac{2}{3}$ root.

3. Extract the cube root of $x^6 - 6x^5 + 40x^3 - 96x - 64$.

LIV–LVI

1. Separate 1522756 into periods of two figures each. How many figures will there be in its square root?

2. How many figures of the square root of 1522756 may be obtained by extracting the square root of 152?

3. Extract the square root of 1522756.

4. How should .00001728 be pointed off for the extraction of the cube root?

5. Extract the cube root of .00001728 true to 4 decimal places.

6. Find the value: $\sqrt{\frac{4}{9}}$; $\sqrt{\frac{50}{72}}$; $\sqrt{\frac{2}{8}}$.

LVII–LXII

1. Find the characteristic of the log: 73865; 73.865; .0073865.

2. Find the mantissa of the log: 73865; 73.865; .0073865.

3. Find the log: 73865; 73.865; .0073865.

4. Find the colog: 73865; 73.865; .0073865.

5. Find the antilog: 3.8765; 0.8765; 8.8765 – 10.

6. Find the value: $.96823 \times 8.64831$; $.83251 \div 6.4372$.

7. Find the value: $\sqrt{762.4} \times \sqrt[3]{5.328}$; $\sqrt[3]{6.254} \div \sqrt{.06297}$.

8. Find the value of x: $1.234^x = 7.235$; $.1234^x = 7.235$.

LXIII–LXIV

1. By factors, express the amount of $100 for 7 yr. 3 mo. 18 da. at 4% compound interest.

2. By factors, express the amount of $100 for 4 yr. 6 mo. 6 da. at 8%, compound interest payable quarterly.

3. Find the amount of $100 for 100 yr. at 5% compound interest.

4. What principal will amount to $1,000,000 in 250 yr. at 6% compound interest?

LXV–LXVII

1. Write with positive exponents: $\frac{x^{-2}y^{-3}}{a^{-1}b^{-2}}$; $\frac{a^2b^2}{x^{-2}y^{-2}}$; $\frac{a^0}{x^{-1}}$.

2. What is the meaning of: $6 \times \frac{2}{3}$? $6 \div \frac{2}{3}$? $16^{\frac{3}{4}}$?

3. Express with radical signs: $x^{\frac{3}{4}}$; $a^{\frac{2}{3}}$; $b^{-\frac{3}{5}}$.

4. Express with fractional exponents: $\sqrt{x^2y}$; $\sqrt[5]{x^3y^4}$; $\sqrt[3]{xy^2}$.

5. Find the value of $16^{\frac{3}{4}}$; $16^{-\frac{3}{4}}$; $\sqrt[3]{27}$; $-\sqrt[3]{27}$.

LXVIII

1. If $27 - 10\sqrt{2} = x - 2\sqrt{xy} + y$, what does $x + y$ equal? what does $-2\sqrt{xy}$ equal?

2. As just suggested, extract the square root of $27 - 10\sqrt{2}$.

LXIX–LXXII

1. Find x: $6x^2 - 5x = -1$. Complete the square by the first method.
2. Find x. Complete the square by the second method.
3. Find x. Use the method of factoring.
4. Find x: $\dfrac{3}{x-5} + \dfrac{2x}{x-3} = 5$.
5. Find x: $x + x^{-1} = 2$.
6. Find x: $\sqrt{2x+8} - 2\sqrt{x+5} = 2$.
7. Find x: $\dfrac{\sqrt{x+3} + \sqrt{x-3}}{\sqrt{3x+3}} = 1$.
8. Find x: $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$.

LXXIII–LXXIV

1. In $x^3 - 6x^2 + 5x + 12 = 0$, how many values has x?
2. Using synthetic division, find by trial the roots of $x^3 - 6x^2 + 5x + 12 = 0$.

LXXV

1. Solve by substitution: $\begin{cases} x^2 + y^2 = 5 \\ xy = 2 \end{cases}$
2. Solve by letting $y = vx$.
3. Solve by letting $x = u + v$ and $y = u - v$.
4. Solve by finding the values of $x + y$ and $x - y$.

LXXVI–LXXXVII

1. If $a : b :: c : d$, what is the result by alternation? by inversion? by composition? by division? by composition and division?
2. If $a : b :: c : d$, prove directly that

$$2a + 3b : 2a - 3b :: 2c + 3d : 2c - 3d.$$

3. Prove indirectly.

4. $x-3:\sqrt{x^2-9}::1:2$. Find x.

5. The radii of two circles are 3 in. and 4 in. respectively. Find the radius of a circle whose area is equivalent to the sum of the areas of the other two. Solve by proportion.

6. Solve by variation.

7. Solve by the usual algebraic method.

LXXXVIII–XC

1. State the fundamental formulæ in arithmetical progression.

2. Given $a=2$, $d=3$, $s=57$. Find n.

3. Given a, d, s. Find n.

4. The sum of three numbers in arithmetical progression is 15; the square of the second exceeds the product of the other two by 4. Find the numbers.

XCI–XCIII

1. State the fundamental formulæ in geometrical progression.

2. Given $a=2$, $r=3$, $s=80$. Find n.

3. Given a, l, s. Find n.

4. The sum of three numbers in geometrical progression is 28; the sum of the first and third exceeds the second by 12. Find the numbers.

XCIV

1. A man bought a farm for \$4212.36, agreeing to pay for it in five equal annual payments, interest at 6%. How much was each payment? Use the U. S. rule for partial payments.

2. A man bought a farm for \$4212.36 and paid \$1000 at the end of each year for 5 yr., interest at 6%. How much did he owe after the fifth payment? Use the U. S. rule.

DEFINITIONS AND INDEX

TERM	P.	DEFINITION
Absolute term	202	A term which does not contain an unknown quantity.
Addends	18	Terms in addition.
Addition	18	Process of uniting numbers into one.
Aggregation	16	Collection into one expression.
Algebra	7	The science of numbers expressed by letters and other symbols, and considered with reference to direction.
Alligation	124	Process of ascertaining cost per unit of mixtures of different ingredients.
Alternation	216	Four quantities are in proportion by alternation when the means are interchanged.
Amount	18	Sum of several numbers; principal plus interest.
Analysis	43	An indirect method of solving problems.
Annuity	243	A fixed sum of money payable annually.
Antecedent	213	First term of a ratio.
Antilogarithm	143	The number corresponding to a logarithm.
Arithmetic	7	The science of numbers expressed by the Arabic notation and without reference to direction.
Arithmetical progression	232	A series of numbers increasing or decreasing by a common difference.
Axiom	243	A self-evident truth.
Base	12	A quantity used several times as an addend, or as a factor.
Binomial	54	An algebraic expression of two terms.
Braces	16	{ }, signs of aggregation.
Brackets	16	[ ], signs of aggregation.
Cancellation	20	Subtraction of a number from its equal; division of both dividend and divisor by the same number.
Characteristic	143	The integral part of a logarithm.
Coefficient	19	The part of a term which shows how many times its unit is used as an addend.
Cologarithm	143	The cologarithm of a number is minus its logarithm.
Comparison	111	The method of elimination of an unknown quantity by placing two expressions of its value equal to each other.

TERM	P.	DEFINITION
Composition	218	Four quantities are in proportion by composition when the sum of the first two terms is to the first or second term as the sum of the last two terms is to the corresponding term.
Compound interest	156	A conception of interest by which the amount at the end of each period bears interest during the next period.
Compound ratio	213	A ratio of a ratio.
Consequent	213	Second term of a ratio.
Continued proportion	214	A proportion in which the consequent of each ratio is the antecedent of the next ratio.
Cube	131	The product of three equal factors.
Cube root	139	One of the three equal factors of a number.
Decimal	144	A fraction written with the aid of a decimal point.
Degree of an equation	94 182 197	The degree of the term which is the highest with reference to unknown literal factors.
Degree of a term	202	The sum of the exponents of the literal factors of a term.
Denominator	76	The part of a fraction which is a divisor; the number showing into how many parts a unit is divided.
Difference	18	Result obtained in subtraction.
Digits	65	The symbols, 1, 2, 3, 4, 5, 6, 7, 8, 9; the numbers denoted by these symbols.
Dividend	29	A number to be divided.
Division	28	Process of finding the other of two numbers when one of them and their product are given.
Divisor	29	A number by which to divide.
Elimination	108	Process of removing an unknown quantity from a system of equations.
Equation	16	Two expressions of the same value connected by '$=$.'
Equimultiples	220	Products of numbers by the same multiplier.
Even number	132	An integer exactly divisible by 2.
Evolution	127	Process of finding a root of a number.
Exponent	14	A number showing how many times the base of a term is taken as a multiplier, or as a divisor.
Extremes	214	First and last terms of a proportion.
Factor	48	An exact divisor of a number.

TERM	P.	DEFINITION
Figures	136	The symbols, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.
Finite number	256	A number that is neither greater nor less than an assignable number.
Formulæ	233	General propositions expressed by letters.
Fraction	76	An expression of division in which the dividend is written above and the divisor below a horizontal line; one or more of the equal parts of a unit.
Geometrical progression	238	A series of numbers which increase or decrease by a common ratio.
Homogeneous equation	202	An equation in which each term except the absolute term is of the same degree with respect to the unknown quantities.
Imaginary	171	An even root of a negative number.
Independent equations	107	Equations which cannot be reduced to the same equation.
Indeterminate equations	121	A set of equations less in number than their unknown quantities.
Index of a root	128	The number which shows what root is to be extracted.
Inequality	16	Two expressions of unequal values connected by '$>$' or '$<$.'
Infinitesimal	256	A number less than any assignable number.
Infinity	256	A number greater than any assignable number.
Integer	121	A whole number.
Inversion	217	Four quantities are in proportion by inversion when the antecedents become consequents and the consequents become antecedents.
Involution	127	Process of finding a power of a number.
Irrational numbers	171	Numbers which cannot be exactly expressed by integers or fractions.
Known quantities	8	Quantities whose values are known at the beginning of a computation.
Limit of a number	257	A value which another number may constantly approach but which it can never reach.
Logarithm	127	The exponent of the power to which a base must be raised to produce a given number.
Mantissa	143	The decimal part of a logarithm.
Mean proportional	214	One of the means of a proportion in which the means are equal.
Means	214	Second and third terms of a proportion.

TERM	P.	DEFINITION
Members	16	Parts of an equation connected by '=.'
Minuend	18	The term from which another is to be subtracted.
Minus	18	The name of the symbol of subtraction.
Mixed number	81	An integer plus a fraction.
Monomial	54	An algebraic expression of one term.
Multiple	68	Number which contains another an integral number of times.
Multiplicand	28	Number to be multiplied.
Multiplication	28	Process of finding the sum when the same number is used several times as an addend.
Multiplier	28	Number by which to multiply.
Negative	14	The opposite of positive; its symbol is '–.'
Notation	7	Art of writing numbers and their relations by symbols.
Number	7	A number is the answer to *how many.*
Numeration	7	Process of naming or reading numbers and their relations.
Numerator	76	The part of a fraction which is the dividend; the number showing how many parts are taken.
Numerical expression	178	An algebraic expression without letters.
Odd number	132	An integer not exactly divisible by 2.
Parentheses	16	( ), signs of aggregation.
Percentage	91	Computations with per cent.
Periods	137	Groups of figures.
Plus	18	The name of the symbol of addition.
Polynomial	22	An algebraic expression of more than one term.
Positive	14	One of two opposite conditions; its symbol is '+.'
Power	128	A product obtained by using the same number several times as a multiplier.
Prime factors	68	Factors that cannot be separated into other factors with integral exponents.
Principle	76	A fundamental truth.
Problem	26	An example in which the operations are not stated.
Product	28	Result in multiplication.
Proof	26	Operation checking accuracy of a calculation.
Proportion	213	An equality of ratios.
Proposition	8	Statement of a truth.

TERM	P.	DEFINITION
Quadratic equation	182	An equation of the second degree.
Quantity	9	Anything to which mathematical processes are applicable.
Quotient	29	Result in division.
Radical sign	172	The symbol '$\sqrt{\ }$' indicating the extraction of a root.
Ratio	213	An expression of division in which the dividend is written before, and the divisor after, a colon.
Rational number	171	A number that can be exactly expressed by an integer or by a fraction.
Real number	246.	A number that is not imaginary.
Reduction	79	Process of changing the form of an expression without changing its value.
Remainder	18	The result in subtraction; the part left undivided in division.
Root	128	One of the equal factors of a number.
Sign	8	A symbol denoting an operation or a relation.
Simultaneous equations	107	Equations which are satisfied by the same values of the unknown quantities.
Sinking fund	243	A sum laid aside at stated intervals for the payment of a debt.
Solution	26	Process by which the answer to a problem is obtained.
Square	48	The product of two equal factors.
Square root	49	One of the two equal factors of a number.
Substitution	112	The putting of an expression in the place of its equal.
Subtraction	18	Process of finding the other of two numbers when one of them and their sum are given.
Subtrahend	18	A number to be subtracted.
Sum	18	The result in addition.
Surd	171	The root of a positive number that cannot be exactly obtained.
Symbol	214	A sign of an operation or of a relation.
Symmetrical equation	202	An equation in which each unknown quantity is used in exactly the same way as each of the others.
Synthetic division	169	Division by the use of coefficients only.
Terms	12	Algebraic expressions consisting of coefficient, base, and exponent; the numerator and denominator of a fraction; nomenclature in the various operations.

TERM	P.	DEFINITION
Theorem	12	A proposition to be proved.
Transposing	94	Transferring a term from one member of an equation to the other.
Type	57	A particular expression made up on the same plan as many other expressions.
Unit	19	All of a term except its coefficient; a single thing.
Unknown quantities	8	Quantities whose values are unknown until the end of a computation.
Variation	226	A ratio whose value always remains the same whatever the values of the terms.
Vinculum	16	$\overline{\quad\quad}$, sign of aggregation.
Whole number	81	An integer or number made up of units which are not fractional.
Zero coefficient	12	A coefficient showing that the base is used the same number of times both as an addend and as a subtrahend.
Zero exponent	14	An exponent showing that the base is used the same number of times both as a multiplier and as a divisor.

ANSWERS

pp. 9 to 13

p. 9 ff.

7. $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$

8. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

9. $\frac{a \times c}{b \times c} = \frac{a}{b}$

10. Multiplying the numerator multiplies a fraction.

11. Multiplying the denominator divides a fraction.

12. Dividing the numerator divides a fraction.

13. Dividing the denominator multiplies a fraction.

14. Multiplying both terms by the same number does not change the value of a fraction.

15. Dividing both terms by the same number does not change the value of a fraction.

24. 20 ; 30 ; 40 ; $5a$

pp. 13 to 15

31. -20 ; -15 ; $-5a$

32. $-5x$; -5×3

34. No

35. That 6 is to be used the same number of times both as an addend and as a subtrahend.

37. 1

38. -1

39. $+1a^{+1}$, or $1a^1$

40. $-4a^6$ means $-a^6-a^6-a^6-a^6$, or that a is used 6 times as a multiplier, and the result 4 times as a subtrahend.

41. 2^5 ; a^5

42. $4a^6$; $-4a^6$

43. 625 ; 125 ; 3125 ; 25

44. 9 ; 18 ; 27

46. $\frac{1}{8}$; $\frac{1}{9}$; $\frac{1}{25}$; $\frac{1}{125}$; $\frac{1}{6}$

47. $\frac{1}{9}$; 1 ; 3

49. No

50. That 6 is to be used the same number of times both as a multiplier and as a divisor ; $6^0=1$.

pp. 16 to 21

51. $8+9=a-b$

52. $8-4>7-5$

53. $8+3<16-2$

54. $6(a-b-c)$

55. $(6a-7b)(a-b)$

56. 36

57. 26

58. 4

59. 18

60. 10

61. 31

62. 50

63. 43

64. 24

65. 108

66. 131

67. 816

68. 507

69. 729

70. 28

71. 32

p. 20 ff.

1. -5 ; 2 ; 5 ; -13 ; -1 ; -2 ; 12

2. -2 ; -11 ; -17 ; 17 ; 1 ; -1 ; 4

4. $6x$; $-3b$; $-y$; $-ab$; $-14xy$; $-10xyz$

pp. 21 to 24

5. $15y$; $-c$; $-10by$; 0; ab; $abcd$
6. $(a-b)x$; $2acy$; $(ac-bd)z$; 0; $-(abc+bcd)x$; 0
7. $3a^2+3b^2$; $-9ab+2$; $2a^2b^2$; $3x^2$
8. $6a^2+ab$
9. $2x^2+2y^2$
10. $4a^2b$
11. $-9x^3-10x^2-2x+13$
12. $11x^3-7x^2-2x-8$
13. $-5a^4+14a^2-17a-4$
14. $-9a^2-6ab+b^2$
15. $7x^2-xy+7y^2$
16. $3x^2+6xy+3y^2$
17. x^4-14a^4
18. $7a^3-3a^2b+6ab^2-7b^3$
19. $27x^2+11y^2$
20. $15a^3+26a^2b+32ab^2+2b^3$
21. -10; 5; -13; -4; 4; 14; 4
22. 12; 16; -16; -3; 11; 5; -4
23. $-7x$; $-3a$; $-15y$; $-2b$; $5x$; $-5abc$
24. y; $-8c$; $-8a$; $18x$; $3ab$; $13bxy$
25. $15x$; $-11y$; $-4b$; $4b$; $-10ab$; $16axy$
27. $4ab$

pp. 24 to 33

28. $2a^2x^2-4abx+2b^2$
29. $-4a^3-3a^2b-2ab^2-5b^3$
30. $-ab-ac-2b^2-2c^2$
31. $6a^3b+2ab^2+8ab^3-3b^4$
32. $8a^4-2a^3+4a^2-15a+14$
33. $abcxy-3aby+5bx-3$
34. 0
37. $(10a-b+8c)x$
38. $(a-b)x-(b-c)y$
39. $(-a-5b)x$
40. $(a-5b+3c)x$
41. $4x^3+9x^2y-4xy^2-3y^3$
42. $4a^4+7a^3-2a-4$
43. $-a^2b^2-18bc-19b^2+1$
44. $-x^4+9x^3y+12x^2y^2+8xy^3-3y^4$

p. 27

4. \$30
5. \$75
6. \$1500
7. 11 cows
8. \$205
9. 110 sheep
10. 25 marbles
11. \$600

p. 30 ff.

1–37. The answers to examples 1 to 37 in multiplication are found in the statements of the corresponding examples in division, and *vice versa.*

pp. 33 to 39

38. 72; 162
39. 200; 2304
40. 4096 or 2^{12}; 32 or 2^5
41. 6561 or 9^4 or 3^8; 243 or 3^5
43. $-7m+38n$
44. $a+4b+3c$
45. $33m^2-38mn+9n^2$
46. b^2-d^2
47. 0
48. $-12a-8b+5c$
49. $6a^2b+6ab^2+2b^3$
51. $8x+y$
52. $11x^2+x$
53. $-5x+5$
54. $2a^3+6ab^2$
55. 0

p. 36 ff.

1–37. The answers to examples 1 to 37 in division are found in the statements of the corresponding examples in multiplication, and *vice versa.*

38. 54; 36
39. 1250; 216 or 6^3
40. 16 or 4^2 or 2^4; 128 or 2^7
41. 3; 27 or 3^3

pp. 40 to 43

43. $6(-x+y); -6(x-y)$
44. $a(b-c)$; $-a(-b+c)$
45. $a(-b+c)$; $-a(b-c)$
46. $3ab(-a+1)$; $-3ab(a-1)$
47. $3bc(b-c)$; $-3bc(-b+c)$
48. $x(-a-b+1)$; $-x(a+b-1)$
49. $ab(1+ab-b^2)$; $-ab(-1-ab+b^2)$
51. $b(x-y)-c(x-y)$
52. $b(x+y)+c(x+y)$
53. $x^2(a+b)+y^2(a+b)$
54. $x^2(a-b)-y^2(a-b)$
55. $x^2(a-b)+y^2(a-b)$
57. $(x-y)(b-c)$
58. $(x+y)(b+c)$
59. $(x^2+y^2)(a+b)$
60. $(x^2-y^2)(a-b)$
61. $(x^2+y^2)(a-b)$
62. $a+b+c$
63. $3a^2-4ab+5b^2$
64. $3a^2+4ab+b^2$
65. $a+x$
66. x^2-a^2
67. $x^3-x^2y-xy^2+y^3$

p. 43 ff.

5. 9 colts
6. 35721
7. Since a dividend is equal to the product of divisor and quotient, no. $=63\times567$.

pp. 43 to 47

8. 7 years
9. 45 women
10. 60 horses
11. 7 dimes, 28 dollars
13. 4 marbles
14. Length of pole, 51 ft.; 1st part, 9 ft.; 2d, 27 ft.; 3d, 15 ft.
15. 32 rods square
17. Since the sum of four consecutive nos. is 94, the sum of the 2 middle nos. is $\frac{1}{2}$ of 94 or 47; the 2 middle nos. are 23 and 24; the others 22 and 25.
18. 25, 29, 33, 37, 41
19. 3, 9, 1, 13
20. 33, 34, 35
21. 20, 25, 30
22. 12, 17
23. \$19
24. 200 pages
25. 28, 32
29. A, \$13; B, \$22
30. Man, 172 lbs.; wife, 143 lbs.
31. A, \$2600; B, \$2400
32. A, 65 yr.; B, 40 yr.
33. \$40
34. A, 44 lbs.; B, 24 lbs.; C, 32 lbs.
35. $\frac{a+b}{2}, \frac{a-b}{2}$
36. A, 21 yr.; B, 9 yr.

pp. 47 to 50

37. A, 40 yr.; B, 30 yr.
38. A, \$110; B, \$50
39. Chain, \$68; watch, \$117
40. 30; 10; 223
41. A, 72 years; B, 24 years
42. 24
43. \$7500
44. \$25
45. Carriage, \$60; horse, \$192

p. 49 ff.

6. $25a^4-20a^2b^2+4b^4$
9. $a^2+b^2+c^2-2ab-2ac+2bc$
10. $a^2+b^2+c^2-2ab+2ac-2bc$
11. $a^2+b^2+c^2+d^2-2ab-2ac-2ad+2bc+2bd+2cd$
12. $a^2+b^2+c^2+d^2+2ab-2ac+2ad-2bc+2bd-2cd$
20. $(7ab-4cd)^2$
21. $(2x^2y-6y^2z)^2$
22. $(5xy+6z)^2$
23. $(a-b-c)^2$
24. $(a+b-c)^2$
25. $(a+b+c)^2$
35. $a^2+2ac+c^2-b^2$
36. $a^2-2ab+b^2-c^2$
37. $a^4+a^2b^2+b^4$
38. a^4+a^2+1
39. x^4+x^2+1
40. $x^4+x^2y^2+y^4$

pp. 51 to 52

45. $(a^2+b^2)(a+b)(a-b)$

46. $(a^3+b^3)(a^3-b^3)$ These can be factored. See p. 60.

47. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$

48. $(a^4+1)(a^2+1)(a+1)(a-1)$

53. $(x+a-b)(x-a+b)$

54. $(a+b+c)(a+b-c)$

55. $(2\,ac+a+b+c)(2\,ac-a-b-c)$

56. $(a+b+c+d)(a+b-c-d)$

57. $(a-b+c-d)(a-b-c+d)$

58. $(x-y+z)(x-y-z)$

59. $(x+z+y)(x+z-y)$

61. $(a+b+c+d)(a+b-c-d)$

62. $(a-b+c-d)(a-b-c+d)$

63. $(a-y+x+z)(a-y-x-z)$

64. $(a+y+x+z)(a+y-x-z)$

66. $(a+b-c)(a-b+c)(a+b+c)(a-b-c)$

67. $(2\,ab+a^2+b^2+c^2)(2\,ab-a^2-b^2-c^2)$

68. $(2\,ab+a^2-b^2-c^2)(2\,ab-a^2+b^2+c^2)$

69. $(a+c+b)(a+c-b)(b+a-c)(b-a+c)$

pp. 52 to 56

71. $(a^2+ab+b^2)(a^2-ab+b^2)$

72. $(1+2\,x^2+2\,x)(1+2\,x^2-2\,x)$

73. $(2\,a^2+6\,ab+3\,b^2)(2\,a^2-6\,ab+3\,b^2)$

74. $(2\,a^2+6\,ab-3\,b^2)(2\,a^2-6\,ab-3\,b^2)$

75. $x^2+3\,x-40$

80. $x^2+xy-6\,y^2$

82. $b^2x^2+2\,bxy-3\,y^2$

83. $4\,x^2-2\,bx-6\,b^2$

84. $9\,a^4+3\,a^2b^2-12\,b^4$

85. $9\,a^2+6\,ab-15\,b^2$

86. $16\,x^4-20\,x^2y^2-14\,y^4$

97. $(ax-8\,b)(ax-2\,b)$

98. $(a-5\,b)(a+2\,b)$

100. $(ab+6\,c)(ab+5\,c)$

101. $(a-6\,bc)(a+5\,bc)$

102. $(1-6\,abc)(1+5\,abc)$

103. $(abc-6)(abc+5)$

104. $(ab+2\,x)(ab+5\,x)$

105. $(a^3+4)(a^3-1)$ See p. 60.

106. $(c^4+20)(c^4-5)$

108. $(x-y+5)(x-y-2)$

109. $a^3+3\,a^2b+3\,ab^2+b^3$

114. $x^{10}+10\,x^9y+45\,x^8y^2+120\,x^7y^3+210\,x^6y^4+252\,x^5y^5+210\,x^4y^6+120\,x^3y^7+45\,x^2y^8+10\,xy^9+y^{10}$

pp. 56 to 60

118. $m^5+10\,m^4n+40\,m^3n^2+80\,m^2n^3+80\,mn^4+32\,n^5$

119. $8\,x^3-36\,x^2y+54\,xy^2-27\,y^3$

120. $a^m+ma^{m-1}b+\frac{m(m-1)}{\lfloor 2}a^{m-2}b^2+\frac{m(m-1)(m-2)}{\lfloor 3}a^{m-3}b^3+\frac{m(m-1)(m-2)(m-3)}{\lfloor 4}a^{m-4}b^4+\cdots$ See p. 28.

122. $(a+b)^3$

123. $(x-y)^7$

124. $(a+1)^6$

125. $(a-b)^5$

126. $(a+1)^8$

127. $(x+y)(x^2-xy+y^2)$

128. $(x-y)(x^2+xy+y^2)$

130. $(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$

131. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$

132. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

133. $(x+y)(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6)$

136. x^3-y^3

137. x^4-y^4

138. x^4-y^4

139. x^6-y^6

140. x^6-y^6

pp. 60 to 61

142. $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$
143. $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$
146. $(2a+3b)(16a^4-24a^3b+36a^2b^2-54ab^3+81b^4)$
147. $(2x-3)(8x^3+12x^2+18x+27)$
148. $(3x-4y)(9x^2+12xy+16y^2)$
150. $(1-m)(1+m+m^2)$
151. $(1+m)(1-m+m^2)$
152. $(1+x^2)(1+x)(1-x)$
154. $(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)$
155. $(x-1)(x^4+x^3+x^2+x+1)$
156. $(x+1)(x^2-x+1)(x-1)(x^2+x+1)$
158. $(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8)$
160. $(x^2+1)(x^4-x^2+1)$
161. $(x^2+1)(x^8-x^6+x^4-x^2+1)$
162. $(x^4+1)(x^8-x^4+1)$
163. $(x^2+y^2)(x^{12}-x^{10}y^2+x^8y^4-x^6y^6+x^4y^8-x^2y^{10}+y^{12})$
164. $(x^2+y^2)(x^{16}-x^{14}y^2+x^{12}y^4-x^{10}y^6+x^8y^8-x^6y^{10}+x^4y^{12}-x^2y^{14}+y^{16})$ or $(x^6+y^6)(x^{12}-x^6y^6+y^{12})$

pp. 61 to 63

165. $(x^4+y^4)(x^{16}-x^{12}y^4+x^8y^8-x^4y^{12}+y^{16})$
166. $7(a-b)(a^2+ab+b^2)$
167. $4(a^2-3ab+4b^2)$
168. $3(a-2b)(a-2b)$
169. $6(a-b)(a-b)$
171. $12(x^2+xy+y^2)(x^2-xy+y^2)$
172. $6(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$
173. $3(x+7)(x-5)$
174. $2a(2x-3y)^2$
175. $10x^2y^2(xy-2y^2+3x^2)$
177. $(x+y)(a-y)$
178. $(x+y)(b-x)$
179. $(bc-x)(ac+x)$
180. $(x-y)(a-b)$
181. $(x^2+y^2)(a+b)(a-b)$
182. $(x+b)(x+a)$
183. $(x+y)(x+z)(x^2-xz+z^2)$
184. $(y+z)(y-z)(x+z)(x-z)$
185. $(a-b)(a-b)(a^2+ab+b^2)$
186. $(3x^2-2y^2)(2a^2-b)$
187. $(y-n)(x+2m)$
189. $9x^{2n}-y^{2n}$
194. $16m^6-36n^2$
198. x^4-6x^2+1
199. $a^{2m+2n}-b^{2m+2n}$
201. a^4-b^4

pp. 63 to 64

202. x^8-y^8
203. a^6-b^6
207. $(x+y)^2-z^2$
208. $(a+b)^2-4(a+b)-21$
209. $(x-y)^2-2z(x-y)-15z$
210. $25x^2-16$
213. a^4-b^4
214. $a^2+b^2+c^2-2ab+2ac-2bc$
215. $a^2+b^2+c^2+d^2-2ab-2ac-2ad+2bc+2bd+2cd$
216. $(x-c)(x+a)$
217. $x^2(x^2-5)(x+1)$
220. $(a+11)^2$
221. $(5a+2b+2c)(5a-2b-2c)$
222. $(y+a)(y-a)(y+b)(y-b)$
223. $(a^2+3a-10)(a^2+3a-4)$ or $(a+5)(a-2)(a+4)(a-1)$
224. $(a^2-b^2+m+n)(a^2-b^2-m-n)$
225. $(a+y)(a+y+2)$
226. $(ab+9c)(ab-5c)$
227. $(a-b+c-d)(a-b-c+d)$
228. $(a-b+n-m)(a-b-n+m)$
229. $(4a+5b)^3$
230. $4an^3x^5(2an^2-3nx^2+ax^3)$

pp. 64 to 69

231. $(m^2+3m-10)(m^2+3m-4)$ or $(m+5)(m-2)(m+4)(m-1)$

232. $(a+b+c-d)(a+b-c+d)$

233. $(x-y+3)(x-y+6)$

234. $x(a-b)(x+y)$

235. $(x+1)^3$

236. $(a+b-c)(a-b)$; multiply and rearrange the terms; a^2-ac-b^2+bc, $a^2-b^2-ac+bc$, etc.

237. $(3x^2+4y^2+8xy)(3x^2+4y^2-8xy)$

238. $(3x^2-4y^2+8xy)(3x^2-4y^2-8xy)$

240. $4x^2y(7x-5)(x+3)$

243. $(x+y)^2(x^2-xy+y^2)^2$

244. $(x^2+x-2)(x^2+x-6)$ or $(x+2)(x-1)(x+3)(x-2)$

245. $(3x^3+1)(1+2y)(1-2y)$

p. 69 ff.

2. $6ax^2$

4. $x-1$

5. a^2bx^2

6. $a-b$

7. $x+4$

8. $4ax^2(a-b)$

9. $a-b$

pp. 69 to 74

10. $x+2$

11. $x-1$

12. $x-3$

13. $x-1$

14. $a-b$

15. $x+2$

16. $a(x-1)$

20. $2x-3$

21. x^2-2x+5

22. a^2-2a+5

23. a^2-a+1

24. $2a-7$

25. $2a-3$

26. $x-2$

27. $3a^2-ax-2x^2$

29. $60x^3y^3z^3$

30. $420a^5x^4y^4$

31. $100x^3y^3z^3$

33. $12(x-3)(x+3)(x-2)$

34. $24a^2x^3(a-x)^4$

35. $x(x-2)^2(x^2-9)$

36 $12(x+y)(x-y)^2(x^2-xy+y^2)$

37. a^3-b^3

39. $12x^4+2ax^3-4a^2x^2-27a^3x-18a^4$

40. $(4a-b)(a-b)(3a^2+b^2)$

41. $2x^4-14x^3+31x^2-31x+12$

42. $(x^3-9x^2+23x-15)(x-7)$

43. $24x^3+22x^2-177x+140$

pp. 74 to 79

44. $(x-3)(3x-4)(5x-2)(x+1)$

p. 75

1. Let ax and bx be the nos. and x their H. C. F. Find their L. C. M. by Prop. XXXVII and by this principle.

6. 6 times. Let $x=$ times A; $\frac{10x}{12}=$ times B; $\frac{10x}{15}=$ times C; the least value of x that makes each a positive integer is 6.

7. The no. of min. before all are together is L. C. M. of 10, 12, and 15 or 60. A goes around 6 times.

p. 79 ff.

1. $\frac{2y}{3xz}$

2. $\frac{2}{3ab^2c^3}$

3. $\frac{a-b}{a+b}$

4. $\frac{a-b}{a+b}$

5. $\frac{a-3}{a+5}$

6. $\frac{a-b}{a^2-ab+b^2}$

7 $\frac{a+b}{a^2+ab+b^2}$

pp. 79 to 80

8. $\dfrac{a^2-ax+x^2}{a+x}$

9. $\dfrac{a^2-b^2}{a^2+b^2}$

10. $\dfrac{x^3-y^3}{x^3+y^3}$

11. $\dfrac{x-5}{x+3}$

12. $\dfrac{2(x+y)}{3(x-y)}$

13. $\dfrac{7(x^2-xy+y^2)}{8(x^4-x^3y+x^2y^2-xy^3+y^4)}$

14. $\dfrac{x^2+y^2}{x^4+x^2y^2+y^4}$

16. $\dfrac{a+b-c}{a-b+c}$

17. $\dfrac{a+b-c}{a+b+c}$

18. $\dfrac{a+b-c-d}{a-b+c-d}$

20. $\dfrac{x-3}{x-5}$

21. $\dfrac{a^2-a-1}{2a^2+a+1}$

22. $\dfrac{5x+2}{7x-4}$

24. $\dfrac{175x^2y^2}{105x^3y^2}$, $\dfrac{42x^3}{105x^3y^2}$, $\dfrac{45y}{105x^3y^2}$

25. $\dfrac{3x+9}{3(x^2-9)}$, $\dfrac{9}{3(x^2-9)}$, $\dfrac{5x-15}{3(x^2-9)}$

pp. 80 to 82

26. $\dfrac{15}{6(1-x^2)}$, $\dfrac{6}{6(1-x^2)}$, $-\dfrac{4+4x}{6(1-x^2)}$

27. $\dfrac{a^2-2ab+b^2}{a^2-b^2}$, $\dfrac{a+b}{a^2-b^2}$, $\dfrac{a^3+ab^2+a^2b+b^3}{a^2-b^2}$

28. $\dfrac{5-10x}{1-4x^2}$, $\dfrac{3x+6x^2}{1-4x^2}$, $\dfrac{4-13x}{1-4x^2}$

30. $\dfrac{(x-2)(x-3)}{(x+4)(x+2)(x-3)}$, $\dfrac{(x+2)^2}{(x+4)(x+2)(x-3)}$

31–36. The answers to Exs. 31 to 36 are
37–42. found in Exs. 37 to 42, and conversely.

44. $-\dfrac{a+3}{24}$

45. $\dfrac{5b^2+4a^2}{120ab}$

46. $\dfrac{53xy}{105}$

47. $\dfrac{y}{12}$

48. $\dfrac{13a}{12c}$

49. $\dfrac{15x+7}{6}$

50. $\dfrac{37ax}{36}$

pp. 82 to 85

51. $\dfrac{25x-61}{56}$

52. $\dfrac{2y-x-z}{30}$

53. $\dfrac{81a-4b}{84}$

54. $\dfrac{4bcd+6acd-3abd-2abc}{48abcd}$

56. $-\dfrac{4m}{m+n}$

57. $\dfrac{1-6x^2}{1-4x^2}$

58. $\dfrac{2x+4a}{x-2a}$

59. $\dfrac{1}{1-9x^2}$

60. $\dfrac{x^2-4xy-y^2}{(x^2-y^2)^2}$

61. $\dfrac{4a}{a+x}$

62. $\dfrac{1}{x^3-1}$

63. $-\dfrac{32a^2}{(1-2a)^2(1+2a)}$

65. 0

66. $\dfrac{y^2}{x^2-y^2}$

67. 0

69. $\dfrac{x-6}{x-3}$

70. $\dfrac{x^2-xy+y^2}{x^2-xy}$

73. a

74. $\frac{1}{2}$

75. $a+b$

pp. 86 to 87

76. $\frac{y(x-12)}{x(y+12)}$

77. $\frac{6a+9}{6a-8}$

78. $\frac{x-y}{x+y}$

79. $x-y$

80. $\frac{m^2+n^2}{-2mn}$

81. $\frac{x+2}{x+4}$

82. $x^4+a^2x^2+a^4$

83. $\frac{2x+3y}{2x-3y}$

84. $5m-\frac{3n}{x}$

85. 1

86. $\frac{x-4}{x+3}$

87. $\frac{-8a+b}{8a}$

88. $\frac{2a^3-2}{a^4+a^2+1}$

or $\frac{2(a-1)}{a^2-a+1}$

89. $\frac{13-18x}{(x+1)(x+2)(x-3)}$

90. 0

91. 0

p. 87 ff.

1. $\frac{x}{2}+\frac{x}{3}=100$

4. 24

5. 60

6. 35 ft.

7. 15

pp. 87 to 91

8. 70

11. $1000

12. F, 40 yr.;
S, 16 yr.

13. 4:48 a.m.

14. $65

15. 42 yr.

16. Wine, 85 gal.; water, 35 gal.

17. 4

18. 75 gal.

19. 4 persons

20. 30 yr.

21. Horse, $480; colt, $280. Let x = value of horse.

22. 840

23. 6

27. Apples, 7¢; eggs, 12¢

28. 9 5¢, 15 2¢, 25 1¢

29. $\frac{1}{2}$¢

30. 12 oranges

31. 16¢. Let x = sell. p. in ¢ of all

32. $\frac{abc}{a-b}$

35. 10%

36. 20%

37. $10,000

38.

$$R=\frac{P}{B};\ R=\frac{60}{150}=40\%$$

39. $B=\frac{P}{R};\ B=\frac{60}{.40}=150$

40. $5\frac{5}{23}\%$

pp. 91 to 93

41. R.R., $11,250; state, $5400. Let x = cost in $ of 1 R. R. share. $1.2x$ = cost in $ of 1 state share. 125 = no. shares R.R. 50 = no. shares state. $125x+60x$ = cost in $ all. $.06\frac{2}{333}\times 185x=1000$.

42. $\frac{100(a-b)}{a}\%$

45. $300

46. $300

47. $200

48. $80

49. $1000

50. $360

51. $400

52. $900

53. $540

55. 6%

56. 3 years

57. 9 mo. 18 da.

58. $125

59. 10%

60. 5%

61. $260

62. 2 yr. 3 mo. 18 da.

63. $\frac{100}{Pt}\%$

64. $\frac{100(A-P)}{Pr}$ years

65. $\frac{100(A-P)}{Pt}\%$

pp. 95 to 98

p. 95 ff.

1. $x=1$
2. $x=-\dfrac{d}{a+b-c}$
3. $x=-\dfrac{3c}{a-b}$
4. $x=\dfrac{a-2c}{b-1}$
6. $x=\dfrac{bc}{a}$
8. $x=12$
9. $x=\dfrac{n+ac}{a+b}$
10. $x=\dfrac{m+ab}{a+c}$
11. $x=-8$
12. $x=2$
14. $x=24$
15. $x=-2$
16. $x=5$
17. $x=3$
18. $x=10$
19. $x=-5$
20. $x=4$
21. $x=11$
22. $x=2\frac{1}{3}$
24. $x=\dfrac{bm-an}{m-n}$
25. $x=-2$
26. $x=-2$
27. $x=2$
28. $x=2$
29. $x=7\frac{76}{107}$
30. $x=-17$
32. $x=2\frac{1}{2}$
33. $x=5$
34. $x=11$

pp. 98 to 101

35. $x=3$
36. $x=8$
38. $x=0$
39. $x=\dfrac{3b}{a}$
40. $x=4$
41. $x=3$
42. $x=2$
43. $x=\frac{2}{5}$
44. $x=4\frac{1}{2}$
45. $x=-\frac{2}{5}$
46. $x=15$
47. $x=8$
48. $x=\dfrac{2m^2}{3n}$
49. $x=10$
50. $x=1$
51. $x=-\dfrac{a^2}{a-b}$
52. $x=-\dfrac{a}{b}$
53. $x=2$
54. $x=6$
55. 8
56. $x=-3$
57. $x=9$
58. $x=4$
59. $x=-7$
60. $x=\dfrac{a+1}{c}$
61. $x=1\frac{1}{2}$
62. $x=3$
63. $x=12$
64. $x=5$
65. $x=30$
66. $x=dm$

pp. 101 to 106

67. $x=\dfrac{a^2(b-a)}{b(b-a)-2a^2}$
68. $\frac{9}{17}$
69. $x=4$
70. $\frac{8}{27}$
71. $x=.2$
72. $a+b$ 73. $9\frac{1}{2}$

p. 102 ff

2. 21 days
3. 18 days
4. Worked, $\dfrac{ac+d}{b+c}$ da.; idle, $\dfrac{ab-d}{b+c}$ da.
6. $11\frac{1}{9}$ hours
7. 75 min.
8. $\dfrac{abc}{ab+ac+bc}$ da.
9. $4\frac{4}{17}$ hours
10. $26\frac{2}{3}$ days
12. 2 mi. per hour
13. $112\frac{1}{2}$ mi.
14. 36 mi. 15. $14\frac{2}{7}$ mi.
16. $\dfrac{abc}{a+b}$ mi.
18. Hound, 220 leaps; fox, 264 leaps
19. No; no
20. $4\frac{1}{2}$ hours
21. $3:16\frac{4}{11}$; $3:32\frac{8}{11}$

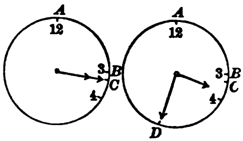
1st

Let $x=$sp. A to C

$\dfrac{x}{12}=$sp. B to C

$15+\dfrac{x}{12}=x$ *Ans.* $3:16\frac{4}{11}$

pp. 106 to 110



2d

Let x=sp. A to D

$\frac{x}{12}$=sp. B to C

$15+\frac{x}{12}+15=x$

Ans. $3:32\frac{8}{11}$

22. $\frac{apr}{p-r}$; 480 mi.

23. $5:46\frac{2}{13}$

24. A, 32 mi.; B, 25 mi.

25. 1 mi. per hour

26. $4:41\frac{5}{11}$

27. First horse, 420 leaps; second horse, 350 leaps

p. 110 ff.

2. $x=3,\ y=2$

3. $x=2,\ z=3$

4. $x=4,\ y=-1$

5. $y=12,\ z=18$

6. $x=-2,\ y=4,\ z=1$

7. $x=5,\ y=-6$

8. $x=-1,\ y=-2$

9. $x=2,\ y=2$

10. $x=16,\ y=12$

11. $x=1,\ y=2,\ z=3$

12. To find 3 equations with 3 unknown quantities, 2 with 2, and 1 with 1.

pp. 110 to 114

13. 1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, or 3 and 4.

14. Any two not already used, but every equation must be used at least once.

17. $y+2z+3w=6 \quad (5)$

$2y-z+4w=5 \quad (6)$

$3y+z+7w=11 \quad (7)$

18. $5z+2w=7 \quad (8)$

$5z+2w=7 \quad (9)$

19. $0=0$; both of the unknown quantities disappear.

20. $x=1,\quad y=1,\quad z=1,\quad w=1$

21. Yes

22. Yes; yes

23. No

24. By addition or subtraction.

25. $x=3,\ y=2$

26. $x=4,\ y=2$

27. $x=2,\ y=1$

28. $x=1,\ y=4$

29. $x=a,\ y=b$

31. $x=\frac{bp}{an+bm}$,

$y=\frac{ap}{an+bm}$

32. $x=\frac{a^2+ab+b^2}{a+b}$, $y=\frac{-ab}{a+b}$

pp. 114 to 118

33. $x=a+b,\ y=a-b$

35. $x=3,\ y=2,\ z=1$

36. $x=2,\ y=1,\ z=0$

37. $x=1,\ y=1,\ z=1$

38. $x=2,\ y=2,\ z=2$

39. $x=6,\ y=7,\ z=8$

41. $x=1,\ y=2,\ z=3$

42. $x=4,\ y=3,\ z=2$

43. $x=38,\ y=20,\ z=5$

44. $x=18,\ y=6$

45. $x=1,\ y=1,\ z=1$

46. $x=3,\ y=2,\ z=1$

p. 117 ff.

4. (*a*) The sum of two numbers is 5; their difference is 1. Find the numbers.

(*c*) Twice the larger of two numbers is their sum plus their difference; twice the smaller is their sum minus their difference.

5. (*b*) Let x = cost 1 calf in \$, then $\frac{89-5x}{6}$ = cost 1 sheep in \$

6. C, 60¢; o., 35¢; r., 65¢

7. Apple, 1¢; pear, 2¢; peach, 3¢; orange, 4¢

pp. 118 to 120

8. Tea, $\dfrac{dm-bn}{ad-bc}$; sugar, $\dfrac{an-cm}{ad-bc}$

9. A's, $5000; B's, $4000; C's, $7000

10. Horse, $252.50; carriage, $200; harness, $47.50

11. Horse, $150; sheep, $5; number of cows, 20

12. 65, 24

13. $180, $120

14. 50 yr.; 30 yr.

15. 5, 2

16. 1st, $\dfrac{a(b-c)}{b-a}$; 2d, $\dfrac{b(c-a)}{b-a}$

17. A, $22; B, $26

18. 12, 48

19. A, 40 da.; B, 60 da.; C, 120 da.

20. A, 50 da.; B, 75 da.

21. $x+10\,y+100\,z$; $z+10\,y+100\,x$

22. 56 **23.** 235

24. 75

25. 523

26. 4337

28. $250

29. $\dfrac{bm-an}{m-n}$ dollars; $\dfrac{100(a-b)}{bm-an}$ %

pp. 123 to 126

32. $x=43, 31, 19, 7$
$y=4, 9, 14, 19$

33. $x=5, 8, 11, \cdots$
$y=4, 12, 20, \cdots$

34. $x=13, 27, 41, \cdots$
$y=2, 5, 8, \cdots$

35. $x=15, 10, 5$
$y=2, 4, 6$

37. $x=22, y=3, z=3$

p. 124 ff.

2. Oats, 4, 8, 12, ...
Bar., 35, 40, 45, ...

3. 9 bu. at $1; 6 bu. at 90¢. The equations are not indeterminate.

4. 2 bu. of blue grass to 1 of clover

5. 10 lb., 4 lb., 6 lb.; 5 lb., 11 lb., 4 lb.

6. 10 gal. water; 20 gal. wine

9. 52, 46, 40, ..., 4
3, 9, 15, ..., 51

10. 505

11. $\frac{55}{28}$, $\frac{20}{63}$

12. 974

13. 12, 11, 10, ..., 1
2, 2, 2, ..., 2
1, 2, 3, ..., 12

14. 3, 1, 29; or 1, 2, 30

15. 9 ways

16. $\frac{3}{2}$, $\frac{4}{5}$, $\frac{5}{8}$, ...

17. P, 5; S, 10; C, 15

18. C, 26, 22, ..., 2
S, 3, 6, ..., 21

pp. 126 to 131

19. H, 11; C, 15

20. C, 1, 2, ..., 16
S, 47, 44, ..., 2
L, 52, 54, ..., 82

21. 3's, 34, 37, 40
5's, 13, 8, 3
8's, 1, 3, 5

22. A, 14, 12, ..., 2
B, 5, 5, ..., 5
C, 1, 2, ..., 7

23. 23, 22, ..., 17
14, 12, ..., 2
3, 6, ..., 21

24. 5's, 1, 2
4's, 3, 1
3's, 1, 2

p. 131

7. A power of the product of factors is equal to the product of their powers; a power of the quotient of two factors is equal to the quotient of their powers.

8-17. See p. 133.

18. In raising to powers, the exponents may be used in any order.

19. $a^3+b^3+c^3+3\,a^2b+3\,a^2c+3\,b^2c+3\,ab^2+3\,ac^2+3\,bc^2+6\,abc$

21. $(a+c)^4-4(a+c)^3b+6(a+c)^2b^2-4(a+c)b^3+b^4$

23. $81\,a^8-732\,a^6b+$

pp. 131 to 135

$864\,a^4b^2-768\,a^2b^3+256\,b^4$

24. $243\,x^{10}-405\,x^8+270\,x^6-90\,x^4+15\,x^2-1$

25. … by the square of each of the terms that follow it, and six times the product of each two terms by each of the others.

p. 133 ff.

7. A root of the product of factors is equal to the product of their roots; a root of the quotient, etc.

8–17. See p. 131.

18. In depressing to roots, the exponents may be used in any order.

19. $a-b$

20. $a+b$

21. $a-b$

22. $2\,x-3\,y$

24. $x-2$

25. $x-2$

26. $2\,x^2-x+1$

28. $x^2-3\,x+2$

29. $11\,x^3-3\,x^2y+5\,xy^2+9\,y^3$

30. x^2-x+1

31. x^2-x+1

pp. 135 to 144

32. No; 6th root

35. 698

36. 1132

37. 231

38. 14

39. 12

40. 248832

41. $\frac{2}{3}$; $\frac{2}{3}$; $\frac{2}{3}$

42. $\sqrt{\frac{6}{9}}$; $\sqrt[3]{\frac{18}{27}}$; $\sqrt[4]{\frac{54}{81}}$

43. 3265; 1,000,000; 6; 6

44. 3 and 265

45. 3 and 265

46. No

48. 3; .014

49. $\sqrt{.4}$; $\sqrt[3]{.666+}$; $\sqrt[4]{.75}$

50. $\frac{1.414+}{1.732+}$, .816+

51. $\sqrt{\frac{6}{9}}$, $\frac{1}{3}\sqrt{6}$, $\frac{1}{3}\times 2.449$, .816+

52. $\sqrt{.666666+}$, .816+

53. $\frac{13}{53}$

54. $\frac{11}{2}$

55. 3.535+

56. .384

57. 6.2

58. 1.2599+

p. 144 ff.

8. 2

9. 1

10. 0

11. 3

12. 2

13. 1

14. 1

15. 2

pp. 144 to 151

16. 3

18. 2364.5

20. 236450000; 23.645

21. 23645; 236450

22. -1 or $9-10$

23. -2 or $8-10$

24. -3 or $7-10$

25. -1 or $9-10$

26. -2 or $8-10$

27. -3 or $7-10$

28. -2 or $8-10$

29. -3 or $7-10$

30. -4 or $6-10$

33. .5114

34. 7.9698 − 10

35. 8.5442 − 10

36. 9.8451 − 10

37. 8.8451 − 10

38. 9.9372 − 10

39. 8.9372 − 10

40. 9.9860 − 10

41. 7.5437 − 10

42. 6.5310 − 10

43. 7.4781 − 10

44. 896.40

45. 3.2464+

47. .03508+

48. .7

49. .07

50. .86540

51. .086540

52. .96825

53. .0034969+

54. .00033961+

55. .0030066+

57. 8.1528 − 10

pp. 151 to 153

58. 9.0785 − 10

59. .4149

60. 9.9046 − 10

61. 8.4371 − 10

62. 0

63. .4219

64. 6.1352 − 10

65. .0620

66. .0322

67. 7.1286 − 10

69. log ( ) = log 26843 + log .38569

70. log ( ) = log 17.846 + log .0036947

71. log ( ) = log .038065 + log .039825

73. log ( ) = log 24789 + colog .017846

74. log ( ) = log .034069 + colog .013758

75. log ( ) = log 632.96 + colog .369481

77. log ( ) = $\frac{2}{5}$ log .73296

78. log ( ) = $\frac{1}{4}$ log .086345

79. log ( ) = $\frac{2}{3}$ log .004839

80. log ( ) = log b + 2 log c + colog d + 2 colog m; log ( ) = 2 log a + 2 log b + 2 colog c + 3 colog d

81. log ( ) = 2 log a + log c + 3 colog b + colog d; log ( ) = 3 log a + 3 log b + colog c + colog d

83. log ( ) = $\frac{1}{2}$(2 log m

pp. 153 to 156

+ 2 log n + log x + 3 colog x + 2 colog y + colog z); log ( ) = $\frac{1}{3}$(4 log a + 2 colog b + 3 colog c)

88. .12626+

89. 4984.4

90. 19.850

91. .36150

92. .000051462+

93. .00031753+

94. 28.386+

95. .005035

97. 3.87+

98. 5.77+

100. −.0868+; with positive mantissa, $\overline{1}.9132$

101. $\dfrac{\log r}{p \log a + q \log b}$

102. $\dfrac{\log a}{2 \log n + \operatorname{colog} m}$

103. −1.111+; with positive mantissa, $\overline{2}.889$

p. 156 ff.

1. $\$(1.06)^{18}$

2. $\$100(1.06)^{18}$

3. $\$750(1.04)^{60}$

4. $\$.01(1.04)^{60}$

6. $\$100(1.06)^{3}(1.03)$

7. $\$100(1.06)^{5}(1.031)$

8. $\$125(1.05)^{7}(1.030\frac{5}{36})$

9. The amount of $100 for 13 yr. 8 mo. 24

pp. 157 to 166

da. at $1\frac{1}{2}\%$ int. comp. annually.

10. The amount of $825.67 for 4 yr. 6 mo. 14 da. at 2% int. comp. annually.

11. $2.8537

12. $285.37

13. $7854

14. $.10471

15. $122.65

16. $137.93+

17. 181.16+

18. $122.80

19. $903.20

25. Because the principal remains the same.

27. $1.14

28. $\$.01(1.06)^{1902} = \$132 \times 10^{44}$

29. $\frac{4}{3} \times 3.1416 \times 92{,}000{,}000^3 \times 5280^3 \times 62\frac{1}{2} \times 19.2 \times 7000 \times \frac{1}{23.2} = \$17384 \times 10^{37}$

30. More than 70,000

p. 163 ff.

1. $\cdots \frac{d}{c^2}$; $\frac{a^3}{b^4}$; $\frac{y^3}{x^2}$; $\frac{x^3}{y^2}$

2. $\cdots a + b$; 1

3. $\cdots \frac{b^3}{a^2}$; $\frac{b^3}{a^4}$; a^3b^4; $\frac{z^3}{xy^2}$; $\frac{mt^b}{ns^a}$

8. $\sqrt[5]{x^2} \times \sqrt[4]{y^3} \times \sqrt{z}$

9. $\sqrt[3]{a^4} \times \sqrt[5]{b^3} \times \sqrt[7]{c^2}$

pp. 166 to 169

10. $\dfrac{\sqrt{m}}{\sqrt[4]{n^3}}$
11. $\sqrt[5]{a^2c^2}\times\sqrt{b}$
12. $\sqrt[n]{x^m}\times\sqrt[m]{y^n}\times\sqrt[n-1]{z^{m-1}}$
14. $a^{\frac{8}{3}}b^{\frac{7}{4}}$
15. $x^{\frac{15}{6}}\times x^{\frac{11}{6}}$
16. $x^{\frac{2}{11}}y^{\frac{3}{10}}$
17. $a^{\frac{m}{n}}b^{\frac{n}{m}}$
18. $a^{\frac{m}{n-1}}b^{\frac{n-1}{m}}$
20. ± 243 ; 16
21. 9 ; -1024
22. ± 243 ; -8
23. 625 ; 36
24. 256 ; 256
25. $\dfrac{2}{3\,a^2b^2x^3y^4}$; $\dfrac{a^4c^4y^4z^5}{x^2b}$
26. $\dfrac{4\,ab}{xy^5}$; $\dfrac{x^4y^3}{5\,m^5n^6}$
27. $\dfrac{3\,b^3c^5}{10\,ad^2}$; $\dfrac{4\,m^2n^4rs}{5}$
28. $\dfrac{m^4s^6}{5\,r^3n^2}$; $\dfrac{3\,a^2d}{4\,c^2b^3}$
29. a^{-1} ; 1 ; x^{-1} ; $a^{-1}b^3c^2$; x
30. $a^{\frac{5}{6}}$; $a^{\frac{1}{4}}$; a^2 ; xy ; $-x^{-\frac{5}{3}}y^{\frac{1}{4}}$; -1
35. a^{-3} ; x^2 ; abc ; a^{-4} ; $-\dfrac{b^3}{a^2c^4}$; $-a^3b$
36. $a^{\frac{1}{2}}$; $a^{\frac{3}{4}}$; $a^2b^{-\frac{1}{2}}$; $x^{\frac{4}{3}}y^{\frac{1}{3}}$; $-x^{-2}y$; $-a^{\frac{1}{2}}b^{\frac{1}{3}}$

pp. 169 to 172

37. $(ab)^{-2}$; 2^{-2} ; $(x+y)^{-3}$; $(a^2-b^2)^{-\frac{1}{2}}$
39. See Ex. 42.
40. See Ex. 43.
42. See Ex. 39.
43. See Ex. 40.
45. Exponents the same.
46. Neither bases nor exponents the same.
47. Either bases or exponents the same.
53. $\sqrt[5]{8}$, $\sqrt{2}$
54. $\sqrt{3}$; $\sqrt{5}$
55. $\sqrt{7}$; $\sqrt[3]{5}$
56. $\sqrt[3]{9}$; $\sqrt{2}$
57. $5\,xy\sqrt{2\,y}$
58. $4\,a^2b^3\sqrt{3\,a}$; $3\,x^3y^3\sqrt{5\,x}$
59. $2\,ab^2\sqrt[3]{2\,a}$; $3\,xy^2\sqrt[3]{3\,y}$
60. $\sqrt{3\,ab}$; $\sqrt[4]{7\,a^2x}$
61. $3\,a^2b\sqrt[3]{2\,ab}$; $3\,x\sqrt{2\,a}$
62. $(a-b)\sqrt{a}$
63. $(a+b)\sqrt{2}$
64. $(a+b)\sqrt{ab}$
65. $2\,a-6\,b$
66. $(2\,x-2\,a)\sqrt{a}$
67. $(a-b)\sqrt[3]{a}$
68. $(a+b)\sqrt[3]{2}$
70. $(x+y)\sqrt[3]{2\,x}$
71. $2\sqrt{x+2\,y}$

pp. 173 to 175

73. $\frac{1}{3}\sqrt{3}$; $\frac{1}{3}\sqrt[3]{9}$
74. $\frac{1}{11}\sqrt{33}$; $\frac{1}{11}\sqrt[3]{3993}$
75. $\frac{1}{6}\sqrt{15}$; $\frac{1}{6}\sqrt[3]{90}$
76. $\frac{1}{9}\sqrt{60}$; $\frac{1}{3}\sqrt[3]{20}$
77. $\frac{1}{6}\sqrt{50}$; $\frac{1}{2}\sqrt[3]{10}$
78. $\frac{1}{2}\sqrt[5]{6}$; $\frac{1}{10}\sqrt{15}$
79. $\frac{1}{3}\sqrt[4]{18}$; $\frac{5}{6}\sqrt[3]{2}$
81. $ab\sqrt{7\,b}$
82. $\dfrac{a-b}{5\,b}\sqrt{15\,b}$
83. $\sqrt{a^2-b^2}$
84. $\frac{1}{6}\sqrt[4]{24\,a}$
85. $\sqrt{x^2-y^2}$
86. $\dfrac{1}{a-x}\sqrt{(a^2-x^2)ax}$
87. $\dfrac{1}{x-y}\sqrt{x^2-y^2}$
89. $3\sqrt{2}$
90. $18\,a^2b^2\sqrt{2\,ab}$
91. $x^3y^3\sqrt{2\,xy}$
92. $11\sqrt{3}$
93. $3\sqrt[3]{5}$
94. $(4\,x+6\,y)\sqrt{3\,x}$
95. $(x-5\,y)\sqrt{3}$
98. $\sqrt{\frac{1}{2}}$; $\sqrt[3]{\frac{1}{3}}$; $\sqrt[4]{\frac{1}{3}}$
99. $\sqrt[3]{1332}$, $\sqrt{125}$
100. $\sqrt[3]{\frac{2}{3}}$, $\sqrt[4]{\frac{1}{2}}$, $\sqrt{\frac{3}{5}}$
102. $(6\,a^3b)^{\frac{5}{6}}$
104. $\sqrt{900\,a^4b^4}$, or $30\,a^2b^2$
105. $\sqrt[3]{625\,x^3y}$, or $5\,x\sqrt[3]{5\,y}$

pp. 175 to 176

106. $7^{\frac{1}{6}}$
107. $(6a^3b)^{\frac{1}{6}}$
108. $\sqrt{9}$, or 3
109. $\sqrt{4a^2b^2}$, or $2ab$
110. $\sqrt[3]{\frac{25y}{x}}$, or $\frac{1}{x}\sqrt[3]{25x^2y}$
111. $(72)^{\frac{1}{6}}$
112. $2048^{\frac{1}{12}}$, or $2^{\frac{11}{12}}$
113. $16\sqrt[6]{675}$
114. $\sqrt[6]{3200}$
115. $\sqrt[6]{72a^7}$, or $a\sqrt[6]{72a}$
116. $\sqrt[6]{\frac{8}{9}}$
117. $\sqrt[12]{32}$
118. $4\sqrt[6]{\frac{27}{25}}$
119. $\sqrt[6]{50}$
122. $\sqrt[3]{750x}$; $\sqrt{48x}$
123. $\sqrt[3]{864a^7}$; $\sqrt[4]{144}$
124. $\sqrt{\frac{1}{a}}$; $\sqrt[3]{2a}$
125. $\sqrt[3]{81a^5}$; $\sqrt[3]{a^4c^3+a^3c^4}$
126. $\sqrt[4]{a^9b}$; $\sqrt{3x^3+6x^2y+3xy^2}$
127. $\sqrt{150x^3y^3}$; $\sqrt{a^{m+2}b^{m+2n}}$
128. $\sqrt{\frac{x^2(x-y)}{x+y}}$; $\sqrt{\frac{x^2(x+y)}{x-y}}$
130. 33

pp. 176 to 178

131. $4a-15\sqrt{ab}-4b$
132. $a^2\sqrt[3]{b^4}-d^2\sqrt[3]{d^4}$, or $a^2b\sqrt[3]{b}-d^3\sqrt[3]{d}$
133. $4-2\sqrt{10}$
135. $\sqrt[3]{9}$; $\sqrt[4]{125}$
136. $\sqrt{2a}$; $\sqrt[3]{4a^2}$
137. $\sqrt{5}-\sqrt{3}$; $\sqrt{5}+\sqrt{3}$
138. $\sqrt{x}-\sqrt{y}$; $\sqrt{x}+\sqrt{y}$
139. $x+\sqrt{y}$; $\sqrt{x}+\sqrt{y}$
141. $\frac{2}{5}\sqrt{5}$; $\frac{2}{5}\sqrt[3]{25}$
142. $\sqrt{2a}$; $\frac{3}{2a}\sqrt{2a}$
143. $\frac{3}{2x^2}\sqrt{2x}$; $\frac{2}{m}\sqrt[3]{5m}$
144. $\frac{5}{3c}\sqrt[3]{3c}$; $\sqrt[4]{2c}$
146. $\frac{2\sqrt{6}-3\sqrt{2}}{3}$
147. $5-2\sqrt{6}$
148. $4+\sqrt{15}$
149. $\frac{x-2\sqrt{xy}+y}{x-y}$
150. 1.414+ ; 1.442+
151. 2d part, 2.884+
152. .702+ ; .480+
153. 2.121+ ; 1.154+
154. .172 ; 5.828+
155. 2d part, .102+
156. 2.236+ ; 1.710+
157. 6.708+ ; 3.420+
158. .447+ ; .584+
159. .894+ ; 1.169
160. .559+ ; .220+

pp. 178 to 187

161. $-.0543+$
162. a^6; a^6; $a^{-8}x^{-6}$; $x^{-6}y^{-6}$; x^4y^6; $a^{-8}b^8c^6d^{-6}$; $(a-b)^{-2}$
163. $a^{\frac{1}{6}}$; $a^{\frac{1}{3}}$; $a^{-\frac{1}{3}}b^{\frac{1}{2}}$; $a^{\frac{1}{30}}$; $a-b$; a
165. $3-\sqrt{2}$
166. $\sqrt{2}+\sqrt{5}$
167. $3-\sqrt{3}$
168. $\sqrt{5}+\sqrt{11}$
169. $\sqrt{2}-\sqrt{3}$
170. $2+\sqrt{-3}$
171. $3-\sqrt{-2}$
172. $2+3\sqrt{5}$

p. 188 ff.

3–48. The answers are the examples of the same numbers pp. 190 to 192.
50. $x=\pm 5$
51. $x=\pm 2a$
52. $x=\pm\sqrt{a+b}$
53. $x=\pm\sqrt{a+b}$
54. $x=\pm\frac{1}{a-b}$
55. $x=1, 2$
56. $x=1, -2$
57. $x=2, 3$
58. $x=a, 2a$
59. $x=a, -2a$
61. $x=\frac{1}{2}, -3$
62. $x=1, -\frac{2}{3}$
63. $x=\frac{2}{3}, -\frac{3}{2}$
64. $x=\frac{a}{2}, -a$

pp. 187 to 191

65. $x=a, \frac{2a}{3}$

74. $x=-\frac{5}{7}, -\frac{1}{3}$

75. $x=\frac{-1\pm 2\sqrt{5}}{3}$

76. $x=5, -4\frac{1}{3}$

77. $x=\frac{3a}{7}, -\frac{a}{3}$

78. $x=a, -\frac{16a}{15}$

82. $x=+1, -1$

84. $x=3, 2$

85. $x=-8, 7$

86. $x=6, -6, 2, -2$

87. $x=7, 2$

88. $x=20, 15$

89. $x=\pm 1, \pm\sqrt{-1}$

91. $x=-4, \pm 1$

92. $x=-1, \pm 2$

93. $x=-2, -3, 1, 2$

94. $x=\pm a, \pm b$

95. $x=\pm a, -a,$
$\frac{a}{2}\pm\frac{a}{2}\sqrt{-3}$

96. $x=-a, -b$

97. $x=a, 2a$

98. $x=-2, -2, -2$

99. $x=\frac{5}{2}, \frac{3}{2}$

100. $x=\pm\sqrt{2}, \pm\frac{1}{2}\sqrt{6}$

p. 190 ff.

3. $x=3, -1$

4. $x=-\frac{14}{3}, 4$

5. $x=\frac{11}{2}, 4$

6. $x=a, \frac{1+a^2}{1-a}$

7. $x=\frac{b}{a}, -\frac{d}{c}$

pp. 191 to 191

8. $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$

9. $x=\frac{1}{4}(1\pm\sqrt{145})$

10. $x=\pm 3\sqrt{2}$

11. $x=1, -\frac{3}{4}$

12. $x=5, 1\frac{1}{5}$

13. $x=2, \frac{32}{5}$

14. $x=7, 2$

15. $y=\pm\frac{1}{2}$

16. $x=\pm 2\sqrt{3}$

17. $y=6, -\frac{9}{2}$

18. $x=\frac{b}{a}, -\frac{a}{b}$

19. $x=2a\pm b$

20. $x=4, -\frac{9}{4}$

21. $x=1, -2$

22. $x=\frac{1\pm\sqrt{17}}{8}$

23. $x=1, -\frac{8}{3}$

24. $x=2, -1$

25. $x=\frac{3\pm\sqrt{21}}{4}$

26. $\frac{-1\pm\sqrt{141}}{14}$

27. $(x^2+12)^{\frac{1}{2}}=2, -3$
$x^2+12=4, 9$
$x=\pm\sqrt{-8}, \pm\sqrt{-3}$

28.
$(3x^2+x-5)^{\frac{1}{2}}=3, -4$
$3x^2+x-5=9, 16$
$36x^2+(\ )+1=169, 253$
$x=2, -\frac{7}{3}, \frac{1}{6}(-1\pm\sqrt{253})$

29. $x=\pm 2, \pm\sqrt{\frac{15}{2}}$

30. $x=11\pm 6\sqrt{3}$

31. $x=\pm 4, \pm 3\sqrt{3}$

pp. 191 to 192

32.
$x^2-7x+18=36, 49$
$x=9, -2, \frac{1}{2}(7\pm\sqrt{173})$

33. $x=\pm 1, \pm 3$

34. $x=1, -2$

35. $x=2, 1, \frac{1\pm\sqrt{5}}{2}$

36. $x=\frac{3}{2}, -\frac{9}{2}, -\frac{1}{2}\sqrt{3},$
$-\frac{5}{2}\sqrt{3}$

37. $x=14\sqrt{26}, 10\sqrt{26}$

38. $x=\pm\sqrt{-28}$

39. $x=4$

40. $x=4, \frac{13}{3}$

41. $x=39, -1$

42. $x=4, \frac{1}{2}$

43. $x=\frac{2\pm\sqrt{3}}{2}$

44. $x=\frac{5\pm\sqrt{697}}{28}$

45. $x=1, -\frac{1}{2}$

46. $x=2, \frac{17}{41}$

47. $x=5, -3, 1$

48. $x=1, -1, 6, -8$

49. $5, -2$

50. $3, -\frac{5}{2}$

51. $1, \frac{1}{2}$

52. $4x^3-(\ )+9=169$
$2x^{\frac{3}{2}}-3=\pm 13$
$x^{\frac{3}{2}}=8, -5$
$x^{\frac{1}{2}}=2, \sqrt[3]{-5}$
$x=4, \sqrt[3]{25}$

53. $5, -1$

54. $4, 1$

55. $\frac{1}{2}, 1$

pp. 192 to 196

56. $5, -1, 2 \pm \sqrt{5}$

57. $0, \pm\sqrt{13}$

p. 194 ff.

4. $\left(\frac{\text{Sum}}{2} + x\right) + \left(\frac{\text{Sum}}{2} - x\right) = \text{Sum}$

5. 14, 6

6. 12.36+ and 7.64+

7. $-a + a\sqrt{5}$ and $3a - a\sqrt{5}$

8. $\pm 91, \pm 7$

9. $\pm\sqrt{ab}$ and $\pm\frac{1}{a}\sqrt{ab}$

10. 60

11. 900 sq. rd.

12. 3 rd.

13. $1\frac{1}{2}$ rd. Observe that the other root, $13\frac{1}{2}$, although positive, does not satisfy the problem.

15. \$80 or \$720

16. \$100

17. \$140

18. \$300

19. \$6

20. $\frac{6000a}{6000 - a}$

21. 24 da.; 18 da.

22. 12 pieces

23. 12 men

24. 20 sheep

25. 25

pp. 196 to 210

26. 18, 10

27. 30 da.

28. 24 da.

29. 70 mi.

30. 25, copper; 2, silver

p. 200 ff.

5. 12

6. 0

7. -6

8. 0

9. 24

10. 72

12. 0

13. 0; 31; -2

14. -47; -44; 1

15. -5; -24; 11

22. $x = 3$

p. 208 ff.

14. The equations are symmetrical; let $x = u + v$ and $y = u - v$

15. Raise (2) to the second power.

17. $x = \pm 2$, $y = \pm 1$; the other values are omitted.

18. $x = \pm 3, \pm 2$; $y = \pm 2, \pm 3$

22. $x = 2, -1$; $y = -1, -2$

27. $x = 2, 1$; $y = 1, 2$

30. $x = 2, 1$; $y = 1, 2$

33. $x = \pm 1$; $y = \pm 2$

35. $x = 2, 3\frac{2}{5}$; $y = 3, -1\frac{1}{5}$

36. $x = \pm 5, \pm 3\sqrt{-1}$; $y = \pm 2, \pm 8\sqrt{-1}$

pp. 210 to 210

37. $x = 3, 2, \frac{5}{2} \pm \frac{1}{2}\sqrt{-151}$; $y = 2, 3, \frac{5}{2} \mp \frac{1}{2}\sqrt{-151}$

38. $x = 3, -2$; $y = 2, -3$

39. $x = 4, -2$; $y = 2, -4$

40. $x = 8, 4$; $y = 4, 8$

41. $x = 4, 2$; $y = 2, 4$

42. $x = 4, -\frac{1}{2}$; $y = 0, -4\frac{1}{2}$

43. $x = 4, 2, \frac{1}{2}(-7 - \sqrt{-35})$; $y = 2, 4, \frac{1}{2}(-7 + \sqrt{-35})$

44. $x = 12, -9$; $y = 9, -12$

45. $x = \pm 4, \pm 3, \pm(\sqrt{22} \pm \sqrt{10})$; $y = \pm 3, \pm 4, \pm(\sqrt{22} \mp \sqrt{10})$

46. $x = 2, 1$; $y = 1, 2$

47. $x = 1, \frac{1}{2}$; $y = \frac{1}{2}, 1$

48. $x = \pm 3$; $y = \pm 1$

49. $x = 6, -2$; $y = 2, -6$

50. $x = \pm 3$; $y = \pm 1$

51. $x = 1, -5$; $y = -1, -7$

52. $x = 2, 1\frac{1}{3}$; $y = 3, 3\frac{2}{3}$

53. $x = 18, 12\frac{1}{2}$; $y = 3, -2\frac{1}{2}$

54. $x = 3$, 3.5844+, $-2.8049+$, $-3.7795+$. The pupil may be excused from finding the decimal parts.

55. $x = 5$; $y = 4$

pp. 210 to 215

56. $x=2, 3, 4$; $y=3, 4, 2$; $z=4, 2, 3$.

p. 211 ff.

2. (a) less $=\dfrac{15-x^2}{x}$

   (b) $15-x^2+\dfrac{(15-x^2)^2}{x^2}=10$

   (c) $\pm 3, \pm 2$

   (d) $xy+y^2=10$
   $xy+x^2=15$

3. 4, 3; 3, 4.
4. 5, -2; 2, -5
5. $\pm 5, \pm 2$
6. 4, -2; 2, -4
7. 3, 2; 2, 3
8. $x+y=8$, $xy=12$
9. ± 8; ± 6
10. 20 S. at $8
11. 400 A., 200 B.
12. 160 yd., 90 yd.
13. 20 ft.; 15 ft.
14. 30 ft.
15. 128 sq. ft., 72 sq. ft., 200 sq. ft.
16. 12 ft., 15 ft.

p. 215 ff.

2. If the product of two quantities is equal to the product of two others, one pair may be made the numerator of one of two equal fractions and the denominator of the other, and the other pair the corresponding terms.

pp. 216 to 222

4. If two fractions are equal, the numerator of one may change places with the denominator of the other.
6. If two fractions are equal, the sum of the terms of the first divided by the numerator or the denominator is equal to the sum of the terms of the second divided by the corresponding term.

16. $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$

$$\frac{a}{b}=\frac{a}{b}$$

$$\frac{a}{b}=\frac{b}{c}$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a^3}{b^3}=\frac{abc}{bcd}=\frac{a}{d}$$

17. $\dfrac{a}{b}=\dfrac{c}{d}$; $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}$;

$a^2-b^2 : a^2 : : c^2-d^2 : c^2$

$$\frac{a^2}{3\,ab}=\frac{c^2}{3\,cd}$$

$a^2-3\,ab : a^2 : : c^2-3\,cd : c^2$

pp. 222 to 225

19. $6\,a^2x$
20. $3:4$ is the greater ratio; by $\frac{3}{16}$
21. $5\,y^2$
22. $\dfrac{x^3}{x^2y+y^3}$
23. That the product of the means shall equal the product of the extremes.
26. $\sqrt{10}$
27. $a+b$, $\dfrac{a-b}{2}$
28. $\dfrac{b(a^2+2\,ab+b^2)}{a^2-2\,ab+b^2}$
29. $3\,a$, $-a$
30. $\dfrac{a(m^3+n^3)}{m^3-n^3}$
31. $\dfrac{a(c^2+1)}{2\,c}$
32. 0, -2
33. 1
34. $\pm\dfrac{a}{b}$

p. 224 ff.

3. 28 ft. $\times$ 21 ft.
4. 8, 4
5. 6, 8
6. 2000 sq. rd.
7. $\dfrac{bnt}{ar}$ yr.
8. B., 440 yd. per min.; T., 352 yd. per min.
9. $\pm 5, \pm 3$
10. $\pm 5, \pm 2$
11. 16 T., 20 T.

pp. 225 to 231

12. A, \$50,000
    B, \$30,000
13. 8 : 7
14. 6, 9, 10, 15

p. 227 ff.

6. $v \propto r^3$
7. $v \propto B \times A$
8. $D \propto R$; R=mi. per hr.
9. $D \propto T$; T=hours
10. $D \propto R \times T$
11. $1 \propto \frac{1}{D^2}$; illumination varies inversely as the square of the distance.
12. $G \propto \frac{W}{D^2}$; force varies directly as the weight and inversely as the square of the distance.

p. 228 ff.

2. 20; $12 : 3 = x : 5$
3. $11\frac{2}{3}$
4. 10; $15 : \frac{1}{4} = 6 : \frac{1}{y}$
5. $A=35$
6. $B=8$
7. $64\,x^2 = 9\,y^3$
9. 2 da.
10. 8 da.
12. 4 wk.
13. 6 men
14. 5 in.
15. 12 in.
16. 9 in.

pp. 231 to 234

17. $15(\sqrt{3}-1)$ in.
18. 576 ft.
19. 64 ft.
20. 32.16
21. 88 da. approx.
22. 250 cu. in.

p. 234 ff.

1. $l=13$
3. $n=7$
4. $d=2$
5. $a=l-(n-1)d$
6. $n=\frac{l-a+d}{d}$
7. $d=\frac{l-a}{n-1}$
8. $l=a+(n-1)d$
9. $s=87$
10. $n=6$
12. $l=22$
13. $n=\frac{2\,s}{a+l}$
14. $a=\frac{2\,s-nl}{n}$
15. $l=\frac{2\,s-an}{n}$
16. $s=\frac{n}{2}(a+l)$
17. $d=5$
18. $d=5$
20. $s=93$
21. $s=93$
22. $s=93$
23. $d=\frac{l^2-a^2}{2\,s-l-a}$
24. $s=\frac{l-a+d}{2\,d}(a+l)$

pp. 235 to 236

25. $l=23$; $s=100$
26. $l=b+2\,an-2\,bn$
    $s=an+an^2-bn^2$
27. $l=(2\,n-1)x$
    $s=n^2x$
28. $l=-\frac{992}{7}$
    $s=-\frac{472500}{7}$
29. $l=-\frac{13}{4}$
    $s=-12$
30. $l=0$
    $s=\frac{n-1}{2}$
31. $n=10$
32. $s=14$
33. $a=-\frac{65}{6}$
34. $n=16$, 11
35. $l=43$
36. $n=8$
37. $14\frac{1}{2}$, 15, $15\frac{1}{2}$; $a=14$, $n=5$, $l=16$
38. 10, 15, 20, 25, 30, 35
39. 16, 20, 24, 28
40. $5\frac{1}{2}$, 10, $14\frac{1}{2}$
41. $\frac{4\,a-7\,b}{3}$, $-\frac{7\,a-4\,b}{3}$
42. a^2+b^2

p. 236 ff.

1. 4, 8, 12
2. 4, 8, 12, 16
3. 3, 6, 9, 12, 15
4. 2, 4, 6, 8
5. 4, 11, 18, …
6. 21
7. 3, 5, 7, … 31
8. 3, 5, 7, …

pp. 237 to 240

9. 16 in., 20 in.
10. 13 yr.
11. \$3486
12. 100
13. 8
14. $m, m+n, m+2n, \cdots$
15. \$1500
16. 852
17. 10 da.
18. 4 da.

p. 240 ff.

1. $l=54$
2. $a=2$
4. $r=3$
5. $a=\frac{l}{r^{n-1}}$
6. $n=\frac{\log l-\log a}{\log r}+1$
7. $r=\sqrt[n-1]{\frac{l}{a}}$
8. $l=ar^{n-1}$
9. $s=75$
10. $r=2$
11. $a=5$
13. $r=\frac{s-a}{s-l}$
14. $a=r(l-s)+s$
15. $l=\frac{s(r-1)+a}{r}$
16. $s=\frac{rl-a}{r-1}$
17. $s=40$
18. $s=40$
19. $s=40$
20. $n=4$

pp. 240 to 241

22. $n=4$
23. $n=\frac{\log l+\operatorname{colog} a}{\log(s-a)+\operatorname{colog}(s-l)}+1$
24. $s=\frac{\left(\sqrt[n-1]{\frac{l}{a}}\right)l-a}{\left(\sqrt[n-1]{\frac{l}{a}}\right)-1}$
25. $l=6561$; $s=9840$
26. $l=\frac{1}{243}$; $s=\frac{1093}{243}$
27. $l=7812.5$; $s=9765.6$
28. $l=a\left(\frac{1}{r}\right)^{n-1}$

    $s=\frac{\frac{a}{r}\left(\frac{1}{r}\right)^{n-1}-a}{\frac{1}{r}-1}$
29. $l=243\sqrt{6}$

    $=595.107+$

    $s=1406+$
30. $l=\frac{8}{45}$; $s=\frac{13}{9}$
31. $r=3$
32. $s=26\frac{2}{3}$
33. $l=2560$
34. $a=4$
35. $s=126$
36. $n=6$
37. 20, 50
38. $-4, 2$
39. 80, 40, 20, 10
40. 6, 12, 24, 48, 96
41. $a(a+b)$, $a(a+b)^2$
42. $6x^2y^2$

pp. 242 to 249

p. 242 ff.

1. 2, 6, 18
2. 1, 2, 4, 8
3. 5, 15, 45
4. 4, 12, 36, 108
5. 3, 9, 15
6. 12, 18, 27
8. \$1234.+
9. \$1977.+
10. \$2958.+
11. \$10,000

12 (*a*) \$557.+

13 (*a*) $31.25(1.00\frac{5}{12})^x=50$; $x=93$

(*c*) $1000(1.00\frac{5}{12})^x$ = amt. 1000

$\frac{24(1.00\frac{5}{12})^x-24}{.00\frac{5}{12}}$ = amt. pay

The sum of these $=5500$ $x=100$

p. 248 ff.

6. $4\sqrt{2}\sqrt{-1}$
7. $2\sqrt{3}\sqrt{-1}$
8. $\sqrt{10}\sqrt{-1}$
9. $9\sqrt{6}\sqrt{-1}$
11. $-15\sqrt{2}\sqrt{-1}$
13. $-\sqrt{-36}$
14. $-\sqrt{-75}$
15. $\sqrt{-72}$
16. $\sqrt{-45}$
17. $\sqrt{-24}$
19. $9\sqrt{-1}$
20. $-\sqrt{2}\sqrt{-1}$

pp. 249 to 257

21. $37\sqrt{2}\sqrt{-1}$

23. $3\sqrt{3}\sqrt{-1}$

24. $\sqrt{5}\sqrt{-1}$

25. $\sqrt{7}\sqrt{-1}$

26–41. The answers to examples 26 to 33 inc. are contained in the data for examples 34 to 41 inc., and *vice versa*.

42. -1

45. $+\sqrt{-1}$

46. -1

p. 253

5. $a^{-\frac{1}{3}}-\frac{1}{3}a^{-\frac{4}{3}}b+\frac{2}{9}a^{-\frac{7}{3}}b^2-\frac{14}{81}a^{-\frac{10}{3}}b^3+\cdots$

6. $a^{\frac{1}{2}}-\frac{1}{2}a^{-\frac{1}{2}}b-\frac{1}{8}a^{-\frac{3}{2}}b^2-\cdots$

7. $a^{\frac{2}{3}}+\frac{2}{3}a^{-\frac{1}{3}}b-\frac{1}{9}a^{-\frac{4}{3}}b^2+\frac{4}{81}a^{-\frac{7}{3}}b^3-\cdots$

8. $a^{-\frac{3}{4}}+\frac{3}{4}a^{-\frac{7}{4}}b+\frac{21}{32}a^{-\frac{11}{4}}b^2+\frac{77}{128}a^{-\frac{15}{4}}b^3+\cdots$

p. 257

1. 0

2. 0

3. 0

4. ∞

5. 0

6. ∞

7. ∞

8. 0

9. ∞

pp. 257 to 261

10. ∞

11. Any no. finite, infinite, or infinitesimal

12. Same as Ex. 11.

13. Same as Ex. 11.

14. Same as Ex. 11.

15. 0

17. $\frac{3}{2}-0$

18. $4-0$

19. $108-0$

20. $\frac{512}{7}-0$

22. $\frac{684}{999}-0$

23 (*a*) 10

(*b*) 20

IX–X

3. $\dfrac{(a+b)^4+(a-b)^4}{(a+b)^2(a-b)^2}$

4. $12\sqrt{3}$;

$\frac{1}{3}\sqrt{2}$;

$-2\sqrt{6}\sqrt{-1}$

5. ∞; ∞; ∞

XI–XII

5. $\sqrt{2}$; $5\sqrt{5}$

6. $\frac{1}{10}\sqrt{5}$

7. $\sqrt{2}\sqrt{-1}$; $-\sqrt{3}\sqrt{-1}$

8. ∞; ∞; any no.

XIII–XVII

10. See p. 248.

11. $18\sqrt{2}$; 8; $12-5\sqrt{6}$

pp. 261 to 264

12. $\sqrt{-9}$; $-\sqrt{-9}$

13. See p. 249.

14. ∞; ∞; any no.

XVIII–XXII

11. -6; 2; $\sqrt{6}-2$

12. $3\sqrt{-1}$; $-3\sqrt{-1}$

13. See p. 250.

14. ∞; 0; any no.; ∞; 0

XXIII

4. $14-6\sqrt{2}+6\sqrt{3}-2\sqrt{6}$

5. $(\sqrt{3}-\sqrt{2})(\sqrt{3}-\sqrt{2})$

6. See Ex. 7.

7. See Ex. 6.

XXIV

10. 1

11. -1

XXVI

3. 2d part:

$x^m-mx^{m-1}y+\dfrac{m(m-1)}{\underline{|2}}x^{m-2}y^2-\cdots$

XXXIII–XXXVI

3. $2x^2-11x+15$ and $6x^2-10x-24$

4. $2x^2-13x+15$ and $10x-15$

5. $x-a$

pp. 264 to 266

XXXVII

3. $(x-y)(4x-y)$
$(3x^2+y^2)$

XLVI–XLIX

9. $x=\frac{9}{10}$, $y=\frac{3}{2}$, $z=\frac{29}{16}$
10. $x=6, 9, 12, \cdots$
$y=2, 4, 6, \cdots$

L–LIII

3. x^2-2x-4

LIV–LVI

4. .000′017 280
6. $\sqrt{\frac{50}{72}}=\sqrt{\frac{25}{36}}=\frac{5}{6}$

pp. 266 to 268

LVII–LXII

8. $x=9.41+$; $x=-.945$

LXIII–LXIV

4. \$.47+

LXIX–LXXII

5. $x=1$
6. $x=\pm 4.$ See p. 247.
7. $x=5, -3.$
8. $x=9, -2,$
$\frac{1}{2}(7\pm\sqrt{173})$

pp. 269 to 269

LXXXVIII–XC

3. $n=$
$$\frac{d-2a\pm\sqrt{8ds+(2a-d)^2}}{2d}$$

XCI–XCIII

3. $n=$
$$\frac{\log l-\log a}{\log(s-a)-\log(s-l)}+1$$

XCIV

1. \$1000.
2. Nothing

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